ISRAEL JOURNAL OF MATHEMATICS, Vol. 27, No. 1, 1977

PRIMARINESS OF SPACES OF CONTINUOUS FUNCTIONS ON ORDINALS

BY
D. ALSPACH[†] AND Y. BENYAMINI[‡]

ABSTRACT
Necessary and sufficient conditions on an ordinal α are given, such that $C(\alpha)$ is primary, and that the general linear group of $C(\alpha)$ is contractible. In particular $C(\alpha)$ possesses both of these properties if α is countable.

Introduction

A Banach space X is said to be primary, if whenever X is isomorphic to $Y \oplus Z$, either Y or Z is isomorphic to X.

Lindenstrauss and Pelczynski [9] have shown that $C(K)$ is primary when K is an uncountable compact metric space. They conjectured that the same holds if K is compact and countable. By a classical theorem of Mazurkiewicz and Sierpinski [11], every compact countable Hausdorff space is homeomorphic to $[1,\alpha]$, for some countable ordinal α, where $[1,\alpha]$ is the space of all ordinals less than or equal to α, endowed with the order topology. We are thus led to study $C(\alpha)$—the space of all continuous functions on $[1,\alpha]$.

Essential to our work is the following complete isomorphic classification of the spaces $C(\alpha)$ (see [1], [8], [16] for earlier partial results). (We identify cardinals with initial ordinals. A cardinal ξ is *regular* if it is not the limit of less than ξ ordinals smaller than ξ.)

CLASSIFICATION THEOREM ([4], [7]). *Let $\alpha < \beta$ be two ordinals of the same cardinality, and let ξ be the first ordinal of this cardinality. Write $\alpha = \xi\eta + \rho$ ($\rho < \xi$). Then there are two cases:*

[†]This is a part of the first author's Ph. D. dissertation, prepared at The Ohio State University under the supervision of Professor W. B. Johnson. The first author was supported by a University Fellowship of The Ohio State University.

[‡]This author's research was partially supported by NSF Grant MPS–74–2449.

Received April 23, 1976 and in revised form October 6, 1976

(1) *If the cardinality of* α *is a uncountable regular cardinal and* $n \leqq \xi$, *then* $C(\alpha)$ *is isomorphic to* $C(\beta)$ *iff when we write* $\beta = \xi\eta_1 + \rho_1 (\rho_1 < \xi)$, *then* η *and* η_1 *have the same cardinality.*

(2) *If* α *is not as above, then* $C(\alpha)$ *is isomorphic to* $C(\beta)$ *iff* $\beta < \alpha^\omega$ (*where* ω *is the first infinite ordinal*).

(If α and β have different cardinalities, $C(\alpha)$ and $C(\beta)$ are certainly not isomorphic.)

We can now formulate our results:

THEOREM 1. *$C(\gamma)$ is not primary iff it is isomorphic to $C(\xi \cdot n)$ where ξ is an uncountable regular cardinal and $1 < n < \omega$.*

In particular if γ is countable we get that $C(\gamma)$ is primary, verifying the above conjecture of Lindenstrauss and Pelczynski. This case was independently proved by Billard [2].

As a by-product of our techniques we also get a necessary and sufficient condition that the general linear group of $C(\alpha)$ is contractible. (See the survey paper [12] for information on contractibility of the general linear group of Banach spaces.)

THEOREM 2. *The general linear group of $C(\alpha)$ is not contractible iff $C(\alpha)$ is isomorphic to $C(\xi \cdot n)$, where ξ is an uncountable regular cardinal and $1 \leqq n < \omega$. In this case $\mathrm{GL}(C(\xi \cdot n))$ has the same homotopy type as $\mathrm{GL}(n, F)$ (where F is the scalar field, real or complex).*

It should be noted that if ξ is an uncountable regular cardinal, then $C(\xi)$ is a primary space whose general linear group is not contractible. This and James' space J seem to be the first examples of such spaces (see [3] for the recent result that J is primary).

We now describe briefly the organization of this paper. In the first section we describe some canonical ways to embed $C(\alpha)$ in $C(\beta)$ and derive some properties of these embeddings. We also give a very simple but useful sufficient condition that an operator on $C(\beta)$ is an isomorphism on some subspace isomorphic to $C(\alpha)$. We end this section by formulating the two basic results used in the proofs of the theorems. The first is a "disjointness lemma" similar to the one proved by Rosenthal [15]. Its proof and another variation of this lemma will be given in the fourth section. The second result is the key to the inductive process and its proof will be given in the last three sections. Assuming these two results we shall prove Theorem 1 in the second section and Theorem 2 in the third section. The reader who is interested only in the separable case may omit the first part of section 5, Lemmas 7.2, 7.3 and the part of the proof of Lemma 7.1 where the case of uncountable cofinality is considered.

We shall use standard notation. Unexplained terms and results on Banach spaces can be found in [10], on topology in [6] and on ordinals and their arithmetic in [17].

If A is a subset of a topological space, we shall denote by $A^{(1)}$ the set of its cluster points. Inductively we define $A^{(\alpha+1)} = (A^{(\alpha)})^{(1)}$ and if α is a limit ordinal $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$. If $A^{(\alpha)}$ is finite and has cardinality n, then (α, n) is called the *characteristic system* of A.

If A is a compact subset of a space of ordinals then the relative topology on A agrees with the order topology on A as an ordered set. In this case there is an α such that $A^{(\alpha)}$ is finite and we denote this α by $\tau(A)$ and call it *the type of* A. If A and B are compact subsets of a space of ordinals, they are homeomorphic iff they have the same characteristic system. Since we shall consider sets of ordinals in their order topology, and this topology is determined by convergence of nets directed over ordinals, we shall use only such nets.

The cofinality of a limit ordinal α is the minimal cardinality of a (nontrivial) net converging to α.

We shall treat real valued continuous functions only. The proofs in the complex case require very minor modifications if any.

§1. Preliminaries

Let A be a closed subset of $[1, \alpha]$. We shall denote by $C(\alpha \mid A)$ the subspace of $C(\alpha)$ of all functions that vanish on A. By $C_0(A)$ we shall denote the space of all continuous functions on A that vanish at the last point of A. In particular, $C_0(\alpha)$ is the space of continuous functions on $[1, \alpha]$ that vanish at α. If $A = [a, b]$ we shall denote $C(A)$ (resp. $C_0(A)$) by $C(a, b)$ (resp. $C_0(a, b)$). The first lemma summarizes some properties of these spaces.

LEMMA 1.1. (a) *For every α there is an isomorphism T of $C(\alpha)$ onto $C_0(\alpha)$ such that* $\|T\|\,\|T^{-1}\| \leqq 3$.

(b) *For every α there is a projection P from $C(\alpha)$ onto $C_0(\alpha)$ with* $\|P\| \leqq 2$.

(c) *For every closed subset A of $[1, \alpha]$ there is a norm one simultaneous extension operator S from $C(A)$ into $C(\alpha)$, such that $S(C_0(A)) \subset C_0(\alpha)$. In particular $C(\alpha) = S(C(A)) \oplus C(\alpha \mid A)$.*

(d) *$C(\alpha \mid A)$ is isometric to $(\Sigma_{a\in A} \oplus C_0(a^-, a))_0 \oplus C(b, \alpha)$, where $b = \sup\{a \in A\} + 1$, and for each $a \in A$, a^- is defined by*

$$a^- = \begin{cases} 1 \textit{ if } a \textit{ is the first element in } A \\ \sup\{b+1\colon b \in A,\ b < a\} \textit{ otherwise}. \end{cases}$$

(Note that if a is a limit point of A, then $a^- = a$ and $C_0(a^-, a)$ is trivial.)

PROOF. The proofs of (a) and (b) are straight-forward and are left to the reader. To prove (c) let $f \in C(A)$ and define for any $\beta \leqq \alpha$, $Sf(\beta) = f(\gamma)$ where

$$\gamma = \begin{cases} \inf\{[\beta,\alpha] \cap A\} & \text{if } [\beta,\alpha] \cap A \neq \emptyset \\ \text{last element in } A & \text{if } [\beta,\alpha] \cap A = \emptyset. \end{cases}$$

The required properties of S are easily checked, and if we denote by $R : C(\alpha) \to C(A)$ the restriction operator, then SR is a projection of $C(\alpha)$ onto $S(C(A))$ whose kernel is exactly $C(\alpha \mid A)$.

To prove (d) notice that if $f \in C(\alpha \mid A)$ and $\varepsilon > 0$, there are at most finitely many $a \in A$ such that $\| f|_{[a^-,a]} \| \geqq \varepsilon$. For otherwise we could find a sequence of points $\alpha_1 < a_1 < \alpha_2 < a_2 < \cdots$ with $a_j \in A$ and $|f(\alpha_j)| \geqq \varepsilon$, but then $\alpha = \sup \alpha_i = \sup a_j$ is in A, since A is closed. Thus $0 = f(\alpha) \neq \lim f(\alpha_j)$ contradicting the continuity of f. Thus every $f \in C(\alpha \mid A)$ can be identified with an element in $(\Sigma_{a \in A} \oplus C_0(a^-, a))_0 \oplus C(b, \alpha)$. The converse is trivial.

To prove Theorem 1 we shall study general linear operators on $C(\gamma)$ for certain types of ordinals γ. We shall find conditions to ensure that an operator $T\colon C(\gamma) \to C(\gamma)$ restricts to an isomorphism on a subspace of $C(\gamma)$ isomorphic to $C(\gamma)$. We shall then apply this result and find that at least one of the projections P and $I - P$ satisfies the required conditions.

The space to which T restricts as an isomorphism is of a very simple type: It is the range of a norm one simultaneous extension operator $S\colon C(A) \to C(\gamma)$ for some subset A of $[1,\gamma]$, homeomorphic to $[1,\gamma]$.

The behaviour of T on $S(C(A))$ is also very simple: For some $\varepsilon > 0$ and a number c, $|TSf(a) - cf(a)| \leqq \varepsilon \|f\|$ for all $f \in C(A)$ and $a \in A$, i.e. up to ε, T acts like multiplication by c on A.

The next definition and lemma will formalize the above.

DEFINITION. Let A be a closed subset of $[1,\alpha]$, and X a closed subspace of $C(\alpha)$. We say that X *is determined by* A if R_A, the restriction operator from $C(\alpha)$ to $C(A)$, is an isometry of X onto $R_A(X)$.

DEFINITION. Let A be a closed subset of $[1,\alpha]$, T a bounded linear operator on $C(\alpha)$, c a real number and $\varepsilon > 0$. We say that a closed subspace X of $C(\alpha)$ is (c,ε) *preserved by* T *over* A, if for all $a \in A$ and $f \in X$, $|(Tf)(a) - cf(a)| \leqq \varepsilon \|f\|$.

The next lemma gives some properties of spaces that are determined or preserved over some set A.

LEMMA 1.2. (a) *Let X be determined over A, and B a closed subset of A. If $R_A(X) = C_0(A)$ (resp. $R_A(X) = C(A)$), then there is a closed subspace Y of X which is determined over B and such that $R_B(Y) = C_0(B)$ (resp. $R_B(Y) = C(B)$).*

(b) *Let X be determined over A and T a bounded linear operator on $C(\alpha)$ such that X is (c, ε)-preserved by T over A, where $c > 3\varepsilon > 0$. Then T is an isomorphism of X onto TX.*

(c) *Let X, A, T, c, ε be as in* (b). *If $R_A(X) = C_0(A)$, then TX is a complemented subspace of $C_0(\alpha)$.*

(d) *Let $A = \overline{\bigcup A_\xi}$ where the A_ξ's are closed and contained in disjoint closed intervals. Let X be determined over A and T a bounded linear operator on $C(\alpha)$ such that X is (c, ε) preserved by T over A where $c > 3\varepsilon > 0$. If $R_A(X) = \{f \in C(\alpha)$: f is supported in A and $f|_{A_\xi} \in C_0(A_\xi)$ for all $\xi\}$ then T is an isomorphism on X and TX is a complemented subspace of $C(\alpha)$, isomorphic to $(\Sigma_\xi \oplus C_0(A_\xi))_0$.*

PROOF. Let $S: C_0(B) \to C_0(A)$ (resp. $S: C(B) \to C(A)$) be a norm one simultaneous extension operator, and define $Y = (R_A|_X)^{-1} S(C_0(B))$ (resp. $Y = (R_A|_X)^{-1} S(C(B))$). This proves (a). To prove (b) let $f \in X$. Since X is determined over A, there is a point $a \in A$ such that $\|f\| = |f(a)|$, then

$$\begin{aligned} \|Tf\| &\geqq |(Tf)(a)| \geqq |cf(a)| - |(Tf)(a) - cf(a)| \\ &\geqq c\|f\| - \varepsilon\|f\| \geqq 2\varepsilon\|f\|. \end{aligned}$$

Since clearly $\|Tf\| \leqq \|T\| \|f\|$, T is an isomorphism on X.

To prove (c), let S be the simultaneous extension operator from $C_0(A)$ to $C_0(\alpha)$ as in Lemma 1.1(c). By the argument of (b) $R_A T|_X$ is an isomorphism of X into $C_0(A)$ which is, as is easy to see, in fact *onto* $C_0(A)$. Thus there is an isomorphism W from $S(C_0(A))$ onto TX such that $WSR_A|_{TX}$ is the identity on TX. Define the projection P from $C(\alpha)$ onto TX by $P = WSR_A$.

(d) follows by the same arguments as (b) and (c).

We are now ready for the two main propositions.

PROPOSITION 1. (The disjointness lemma) *Let T be a bounded linear operator on $C(\alpha)$. Let $\{X_\beta\}_{\beta \in \mathfrak{B}}$ and $\{A_\beta\}_{\beta \in \mathfrak{B}}$ satisfy:*

(a) *X_β is a closed subspace of $C(\alpha)$, A_β is a closed subset of $[1, \alpha]$.*

(b) *If $\gamma \neq \beta$, $f_\gamma \in X_\gamma$, $f_\beta \in X_\beta$, then f_γ and f_β have disjoint supports.*

Then for every $\varepsilon > 0$, there is a subset $\mathscr{B}_1$ of $\mathscr{B}$ of the same cardinality as $\mathscr{B}$, and sets $\{B_\beta\}_{\beta \in \mathscr{B}_1}$ such that:

(1) *For each $\beta \in \mathscr{B}_1$, B_β is a closed subset of A_β and $\tau(B_\beta) = \tau(A_\beta)$.*

(2) *For each $\beta \in \mathscr{B}_1$, $a \in B_\beta$ and function $f \in \operatorname{sp}\{X_\gamma : \gamma \in \mathscr{B}_1, \gamma \neq \beta\}$, we have $|Tf(a)| \leqq \varepsilon \|f\|$.*

PROPOSITION 2. *Let α be either a successor or a regular uncountable cardinal, and let T be a bounded linear operator on $C_0(\omega^{(\omega^\alpha)})$. Then for every $\varepsilon > 0$ there exist a closed subset A of $[1, \omega^{(\omega^\alpha)}]$ homeomorphic to $[1, \omega^{(\omega^\alpha)}]$, a closed subspace X of $C_0(\omega^{(\omega^\alpha)})$ and a number c, such that*

(a) *X is determined over A and $R_A(X) = C_0(A)$.*

(b) *X is (c, ε)-preserved by T over A.*

The proof of Proposition 1 will be given in the fourth section, and the last three sections are devoted to the proof of Proposition 2.

§2. Proof of Theorem 1

We start by noticing that if ξ is an uncountable regular cardinal and $n > 1$ then $C(\xi \cdot n) = C(\xi) \oplus C(\xi(n-1))$ and by the classification theorem (see Introduction) neither of those is isomorphic to $C(\xi \cdot n)$. Thus $C(\xi \cdot n)$ is not primary.

By the classification theorem there are four cases remaining to be considered:

Case I. $\gamma = \omega^{(\omega^\alpha)}$ where $\alpha = \beta + 1$.

Case II. $\gamma = \omega^{(\omega^\alpha)}$ where α is an uncountable regular cardinal (in this case $\gamma = \alpha$).

Case III. $\gamma = \xi \cdot \lambda$ where ξ and λ are infinite cardinals, $\lambda \leqq \xi$ and ξ is uncountable and regular.

Case IV. $\gamma = \omega^{(\omega^\alpha)}$ where α is a limit ordinal which is not a regular uncountable cardinal. (This last case also includes the case when α is an uncountable cardinal which is not regular. For such α, $\omega^{(\omega^\alpha)} = \alpha$.)

The first two cases are exactly those covered by Proposition 2, and we start by considering them simultaneously. Thus assume that $C_0(\omega^{(\omega^\alpha)}) = Z \oplus Y$ with P_Z, P_Y the projections. By Proposition 2 there is a subspace X of $C_0(\omega^{(\omega^\alpha)})$, a subset A of $[1, \omega^{(\omega^\alpha)}]$, homeomorphic to $[1, \omega^{(\omega^\alpha)}]$ and a number c such that X is determined over A, $R_A(X) = C_0(A)$ and X is $(c, 1/10)$-preserved by P_Z over A. Since $P_Z + P_Y = I$, X is $(1-c, 1/10)$-preserved by P_Y over A, and we can thus assume that $c \geqq 1/2$. But $1/2 \geqq 3 \cdot 1/10$ and thus by Lemma 1.2(b) and (c), Z

contains a complemented copy of $C_0(\omega^{(\omega^\alpha)})$. We now distinguish between the two cases.

Case I. By the classification theorem, we get that in this case $C_0(\omega^{(\omega^\alpha)})$ is isomorphic to its c_0-sum, i.e. $(\Sigma \oplus C_0(\omega^{(\omega^\alpha)}))_0$. Since Z is complemented in, and contains a complemented copy of $C_0(\omega^{(\omega^\alpha)})$, it is in fact isomorphic to $C_0(\omega^{(\omega^\alpha)})$ by Pelczynski's decomposition method [14].

Case II. We cannot apply here the decomposition method directly because $C_0(\alpha)$ is not isomorphic to its c_0-sum (not even to its square). We thus study the projection from $C_0(\alpha)$ onto $P_Z X$ more carefully. We shall use here Lemmas 1.1 and 1.2 and keep the same notation. Without loss of generality we can assume that $\sup\{\tau([a^-, a]) : a \in A\} = \alpha$ (where $\tau(B)$, the type of the set B is the last β such that $B^{(\beta)} \neq \emptyset$). Indeed if this is not the case, we can find a subset A_1 of A, still homeomorphic to $[1, \alpha]$ for which the above condition holds. We then use Lemma 1.2(a) to replace X by a subspace X_1 of X which is determined by A_1 and $R_{A_1}(X_1) = C_0(A_1)$. By Lemma 1.1(d) $C_0(\alpha \mid A)$ is isomorphic to $(\Sigma_{a\in A} \oplus C_0(a^-, a))_0$ which is isometric to $(\Sigma_{a\in A} \oplus C_0(\alpha_a))_0$, where the net $\{\alpha_a\}$ is unbounded in α (by our condition). Using the decomposition method, it is easy to check that under this condition $(\Sigma_{a\in A} \oplus C_0(\alpha_a))_0$ is isomorphic to its c_0-sum. Thus by the decomposition method again, if Z_1 is a complemented subspace of $C_0(\alpha \mid A)$, then $C_0(\alpha \mid A) \simeq C_0(\alpha \mid A) \oplus Z_1$.

By the proof of Lemma 1.2(c), the kernel of the projection from $C_0(\alpha)$ onto $P_Z X$ is exactly $C_0(\alpha \mid A)$, and thus $Z \simeq C_0(\alpha) \oplus Z_1$ where Z_1 is a complemented subspace of $C_0(\alpha \mid A)$. Using the above remark and the fact that $C_0(\alpha) \simeq C_0(A) \oplus C_0(\alpha \mid A)$ (Lemma 1.1(c)) we get that

$$
\begin{aligned}
Z &\simeq C_0(\alpha) \oplus Z_1 \simeq (C_0(A) \oplus C_0(\alpha \mid A)) \oplus Z_1 \\
&\simeq C_0(A) \oplus (C_0(\alpha \mid A) \oplus Z_1) \simeq C_0(A) \oplus C_0(\alpha \mid A) \simeq C_0(\alpha).
\end{aligned}
$$

The proofs of Cases III and IV are very similar, so that we prove only Case III in detail. We then indicate how to prove Case IV.

Case III. For each $\delta < \lambda$, let $D_\delta = [\xi\delta + 1, \xi(\delta + 1)]$, and let $D = \{\xi\delta : \delta \leqq \lambda\}$. By Lemma 1.1(c) and (d), $C_0(\xi \cdot \lambda) \simeq (\Sigma_{\delta<\lambda} \oplus C_0(D_\delta))_0 \oplus C_0(D)$. But $C_0(D)$ is isometric to $C_0(\lambda)$ and each $C_0(D_\delta)$ is isometric to $C_0(\xi)$. We claim that $C_0(\xi \cdot \lambda)$ is isomorphic to $(\Sigma_{\delta<\lambda} \oplus C_0(\xi))_0$. Indeed, since $\lambda \leqq \xi$ write $C_0(\xi) \simeq W \oplus C_0(\lambda)$, and then

$$C_0(\xi\cdot\lambda)\simeq\Big(\sum_{\delta<\lambda}\oplus C_0(\xi)\Big)_0\oplus C_0(\lambda)$$
$$\simeq\Big(\sum_{\delta<\lambda}\oplus W\Big)_0\oplus\Big(\sum_{\delta<\lambda}\oplus C_0(\lambda)\Big)_0\oplus C_0(\lambda)$$
$$\simeq\Big(\sum_{\delta<\lambda}\oplus W\Big)_0\oplus\Big(\sum_{\delta<\lambda}\oplus C_0(\lambda)\Big)_0\simeq\Big(\sum_{\delta<\lambda}\oplus C_0(\xi)\Big)_0.$$

Assume now that $C_0(\xi\cdot\lambda)=Z\oplus Y$, and let P_Z, P_Y be the projections. For each $\delta<\lambda$ let R_δ: $C(\xi\cdot\lambda)\to C(D_\delta)$ be the restriction operator, and identify $C(D_\delta)$ with the space of all continuous functions on $[1,\xi\cdot\lambda]$ that vanish off D_δ. The operator $T_\delta=R_\delta P_Y|_{C(D_\delta)}$ is thus a bounded linear operator on $C(D_\delta)$. Since $C(D_\delta)$ is isometric to $C(\xi)$ and ξ is an uncountable regular cardinal, we can apply Proposition 2, to find a closed subset A_δ of D_δ homeomorphic to $[1,\xi]$, a subspace X_δ of $C(D_\delta)$ and a number c_δ such that

(1) X_δ is determined over A_δ and $R_{A_\delta}(X_\delta)=C_0(A_\delta)$,

(2) X_δ is $(c_\delta,1/40)$-preserved by T_δ over A_δ.

We can now find a subset $\mathcal{B}$ of $\{\delta\colon\delta<\lambda\}$, of the same cardinality as λ, and a number c such that $|c_\delta-c|<1/40$ for all $\delta\in\mathcal{B}$. Clearly then X_δ is $(c,1/20)$-preserved by T_δ over A_δ for all $\delta\in\mathcal{B}$. We also note that if we define $S_\delta=R_\delta P_Z|_{C(D_\delta)}$, then $S_\delta+T_\delta$ is the identity on $C(D_\delta)$, and thus for each $\delta\in\mathcal{B}$, X_δ is $(1-c,1/20)$-preserved by S_δ over A_δ. We can thus assume that $c\geqq 1/2$.

We now use the disjointness lemma (Proposition 1) for the operator P_Y, $\{X_\delta\}_{\delta\in\mathcal{B}}$, $\{A_\delta\}_{\delta\in\mathcal{B}}$, and $\varepsilon=1/20$, to find a subset $\mathcal{B}_1$ of $\mathcal{B}$ of the same cardinality, and subsets B_δ of A_δ satisfying (1) and (2) of Proposition 1. By Lemma 1.2(a), we can find for each $\delta\in\mathcal{B}_1$ a subspace Y_δ of X_δ which is determined over B_δ and $R_{B_\delta}(Y_\delta)=C_0(B_\delta)$.

Let $W=\overline{\mathrm{sp}}\{Y_\delta\colon\delta\in\mathcal{B}_1\}$, $E=\overline{\bigcup_{\delta\in\mathcal{B}_1}B_\delta}$. We shall show that $P_Y|_W$ is an isomorphism of W onto a complemented subspace of $C(\xi\cdot\lambda)$ isomorphic to $(\Sigma_{\delta\in\mathcal{B}_1}\oplus C_0(B_\delta))_0$. By the remarks in the beginning of the proof this last space is isomorphic to $C_0(\xi\cdot\lambda)$, and thus Y is isomorphic to $C_0(\xi\cdot\lambda)$ by the decomposition method.

W is $(c,1/10)$-preserved by P_Y over E. Indeed, if $f\in W$ and $b\in B_\delta$ for some $\delta\in\mathcal{B}_1$, f has a unique representation as $g+h$ where $g\in Y_\delta$, $h\in\overline{\mathrm{sp}}\{Y_\gamma\colon\gamma\in\mathcal{B}_1,\ \gamma\neq\delta\}$, and $\|g\|,\|h\|\leqq\|f\|$. Then

$$|P_Yf(b)-cf(b)|\leqq|P_Yg(b)-cg(b)|+|P_Yh(b)-ch(b)|$$
$$\leqq\frac{1}{20}\|f\|+\frac{1}{20}\|f\|\leqq\frac{1}{10}\|f\|.$$

The first estimate follows from $g \in Y_\delta$, $b \in B_\delta$ and the fact that Y_δ is $(c, 1/20)$-preserved by P_Y over B_δ. The disjointness condition and $h(b) = 0$ give

$$|P_Y h(b) - ch(b)| \leqq \frac{1}{20}\| h \| \leqq \frac{1}{20}\| f \|.$$

Since $c \geqq 1/2 \geqq 3 \cdot 1/10$, we get by Lemma 1.2(d) that indeed $P_Y|_W$ is an isomorphism onto a complemented subspace of $C_0(\xi \cdot \lambda)$ isomorphic to $(\Sigma_{\delta \in \mathcal{B}} \oplus C_0(B_\delta))_0$.

Case IV. Since α is a limit ordinal, let λ be its cofinality and choose a net $\{\alpha_\delta : \delta < \lambda\}$ such that $\alpha_\delta < \alpha_{\delta+1}$, α_δ are non-limit ordinals and $\alpha_\delta \uparrow \alpha$. Let $D_\delta = [\omega^{(\omega^{\alpha_\delta})} + 1, \omega^{(\omega^{\alpha_{\delta+1}})}]$.

Again by Lemma 1.1(c) and (d) and the decomposition method, $C_0(\omega^{(\omega^\alpha)}) \simeq (\Sigma_{\delta<\lambda} \oplus C_0(D_\delta))_0$. Assume $C_0(\omega^{(\omega^\alpha)}) = Z \oplus Y$. By Proposition 2, we can find for each $\delta < \lambda$, a subset A_δ homeomorphic to D_δ, subspace X_δ of $C_0(D_\delta)$, and number c_δ, such that X_δ is determined over A_δ, $R_{A_\delta}(X_\delta) = C_0(A_\delta)$, and X_δ is $(c_\delta, 1/40)$-preserved by $R_{D_\delta} P_Y|_{C(D_\delta)}$ over A_δ. Again by passing to a subset $\mathcal{B}$ of $\{\delta : \delta < \lambda\}$ of the same cardinality as λ, we can assume that there is a c such that each X_δ is $(c, 1/20)$-preserved, and also that $c \geqq 1/2$. We next use the disjointness lemma and finish in the same way as in Case III.

§3. Proof of Theorem 2

Let ξ be an uncountable regular cardinal and $1 \leqq n < \omega$. We shall briefly describe how the proof of the classification theorem (see Introduction) shows that the general linear group of $C(\xi \cdot n)$ is not contractible. This follows along the same lines as the case $\xi = \omega_1$ (the first uncountable ordinal) which was derived in [13] from the results in [16].

Fix an uncountable regular ξ, and let $X_\xi \subset C(\xi)^{**}$ be the subspace of $C(\xi)^{**}$, of all functionals φ with the following property: If $\{\mu_\beta\}_{\beta<\alpha}$ is a bounded net in $C(\xi)^*$ which is ω^*-convergent to zero, and if $\alpha < \xi$, then $\varphi(\mu_\beta) \to 0$.

It was shown in [7] that X_ξ is a closed subspace of $C(\xi)^{**}$, containing $C(\xi)$ as a subspace of co-dimension one. Now if T is any operator on $C(\xi)$, T^{**} takes X_ξ into itself, and extends T, and thus induces in a natural way an operator $\hat{T}$ on the one-dimensional space $X_\xi / C(\xi)$. It is clear that if T is invertible, so is $\hat{T}$.

If $n > 1$, $C(\xi \cdot n) = C(\xi) \oplus \cdots \oplus C(\xi)$ (n-times), and if T is an invertible operator on $C(\xi \cdot n)$, it naturally induces an invertible operator $\hat{T}$ on the n-dimensional space $X_\xi / C(\xi) \oplus \cdots \oplus X_\xi / C(\xi) = F^{(n)}$ (where F is the real or complex field). By the same argument as the one given in [13] for $\xi = \omega_1$, it can be shown that $\mathrm{GL}(C(\xi \cdot n)) = G_1 \times \mathrm{GL}(n, F)$ where $G_1 = \{T \in \mathrm{GL}(C(\xi \cdot n)) \colon \hat{T}$

is the identity on $F^{(n)}\}$ is contractible. Thus $\mathrm{GL}(C(\xi\cdot n))$ has the same homotopy type as $\mathrm{GL}(n,F)$.

We now pass to the positive part of Theorem 2. For background the reader is referred to [12]. As a very special case of the results in [12] we quote the following sufficient criterion:

Let X be a Banach space with the following two properties:

(1) *X is isomorphic to its c_0-sum.*

(2) *If $T_1,\cdots,T_n$ are bounded linear operators on X, and $\varepsilon>0$, there are two norm one projections P_1, P_2 on X such that P_1X and P_2X are isomorphic to X, and such that $\|P_1T_jP_2\|<\varepsilon$ for all $1\leqq j\leqq n$.*

Then the general linear group of X is contractible.

To apply this criterion to our situation, we first notice that by the classification theorem, unless $C(\alpha)$ is isomorphic to $C(\xi\cdot n)$ with ξ uncountable regular cardinal and n an integer, $C(\alpha)$ is isomorphic to its c_0-sum, which is just $C_0(\alpha\cdot\omega)$. Thus the first condition of the criterion holds. To show that the second condition holds, we represent $C(\alpha)$ as $C_0(\alpha\cdot\omega)$ and put $A_m=[\alpha m+1,\alpha(m+1)]$. Given an $\varepsilon>0$ and finite number of operators $T_1,\cdots,T_n$ on $C_0(\alpha\cdot\omega)$, we use the disjointness lemma (Proposition 1) inductively n times, to find an infinite set M of natural numbers, and for each $m\in M$ a subset B_m of A_m homeomorphic to $[1,\alpha]$, such that for each $m_0\in M$, $b\in B_{m_0}$ and f which is supported in $\bigcup\{A_m: m\in M, m\neq m_0\}$, we have $|T_jf(b)|\leqq\varepsilon\|f\|$ for $j=1,\cdots,n$.

For each $m\in M$, let $S_m: C(B_m)\to C(A_m)$ be a norm one simultaneous extension operator. Write now $M=M_1\cup M_2$, a disjoint union of infinite sets, and define P_1, P_2 by

$$P_if(a)=\begin{cases} S_m(f|_{B_m})(a) & \text{if } a\in A_m,\ m\in M_i \\ 0 & \text{otherwise.}\end{cases}$$

Clearly P_1, P_2 are norm one projections in $C_0(\alpha\cdot\omega)$ whose ranges are isometric to $C_0(\alpha\cdot\omega)$, and by the choice of M, we get that $\|P_1T_jP_2\|<\varepsilon$, $j=1,\cdots,n$.

§4. The disjointness lemma

We start this section by proving the "disjointness lemma" (Proposition 1). We then give another variation which will be used in the proof of Proposition 2 in the sixth section. The proof that we give is similar to Kupka's proof [5] of Rosenthal's disjointness lemma [15].

Proof of Proposition 1. Since T is a bounded linear operator, the condition (2) is already satisfied for $\varepsilon=\|T\|$ and $\mathcal{B}_1=\mathcal{B}$, $B_\beta=A_\beta$. It is thus enough to show that if (X_β, A_β) satisfy (a), (b) and the condition (2) for some $\varepsilon > 0$, we can find $\mathcal{B}_1$ and $\{B_\beta\}_{\beta\in\mathcal{B}_1}$ that will satisfy (1) and the condition (2) for $\varepsilon/2$. The general case follows by a finite number of iterations of this process.

Re-index the family (X_β, A_β) as $(X_{(\beta,\gamma)}, A_{(\beta,\gamma)})_{\beta,\gamma\in\mathcal{C}}$ where $\mathcal{C}$ is an index set of the same cardinality as $\mathcal{B}$. For each $\beta,\gamma\in\mathcal{C}$ define $B_{(\beta,\gamma)} = \{a \in A_{(\beta,\gamma)}: \sup|(Tf)(a)| \leqq \frac{1}{2}\varepsilon\|f\|\}$, where the sup is taken over all f in $\mathrm{sp}\{X_{(\beta,\delta)}: \delta\in\mathcal{C}, \delta\neq\gamma\}$. $B_{(\beta,\gamma)}$ is a closed subset of $A_{(\beta,\gamma)}$ and there are two possibilities:

(I) If there is a $\bar\beta$ such that $\tau(B_{(\bar\beta,\gamma)}) = \tau(A_{(\bar\beta,\gamma)})$ for all $\gamma\in\mathcal{C}$, we are done by taking $\mathcal{B}_1 = \{(\bar\beta,\gamma): \gamma\in\mathcal{C}\}$ and $B_{(\bar\beta,\gamma)}$.

(II) If no such $\bar\beta$ exists we can choose for each $\beta\in\mathcal{C}$ a $\gamma(\beta)\in\mathcal{C}$ with $\tau(B_{(\beta,\gamma(\beta))}) < \tau(A_{(\beta,\gamma(\beta))})$. We now take $\mathcal{B}_1 = \{(\beta,\gamma(\beta)): \beta\in\mathcal{C}\}$ and let $B_\beta = \{a\in A_{(\beta,\gamma(\beta))}: \sup|Tf(a)| \leqq \frac{1}{2}\varepsilon\|f\|\}$, where the sup is taken over all f in $\mathrm{sp}\{X_{(\beta_1,\gamma(\beta_1))}: \beta_1\in\mathcal{C}, \beta_1\neq\beta\}$.

Then B_β is a closed subset of $A_{(\beta,\gamma(\beta))}$ and clearly condition (2) is satisfied for $\varepsilon/2$. It only remains to show that $\tau(B_\beta) = \tau(A_{(\beta,\gamma(\beta))})$ for all $\beta\in\mathcal{C}$. By the choice of $\gamma(\beta)$ if this were not the case then $B_\beta \cup B_{(\beta,\gamma(\beta))} \subsetneqq A_{(\beta,\gamma(\beta))}$. (It is clear that if B, C are subsets of A with $\tau(B), \tau(C) < \tau(A)$ then also $\tau(B\cup C) < \tau(A)$ hence $B\cup C \subsetneqq A$.) Let $a\in A_{(\beta,\gamma(\beta))}$ be a point not in $B_{(\beta,\gamma(\beta))}\cup B_\beta$, then by the definition of these sets there are functions

$$f\in \mathrm{sp}\{X_{(\beta,\gamma)}: \gamma\neq\gamma(\beta)\}, \qquad \|f\| = 1$$

$$g\in \mathrm{sp}\{X_{(\beta_1,\gamma(\beta_1))}: \beta_1\neq\beta\}, \qquad \|g\| = 1$$

such that $Tf(a) > \varepsilon/2$, $Tg(a) > \varepsilon/2$. But by our choice of indices and the condition (b) of the lemma f and g have disjoint supports, and thus $\|f+g\| = 1$. This contradicts the assumption that (2) is satisfied with ε since

$$T(f+g)(a) = Tf(a) + Tg(a) > \varepsilon = \varepsilon\|f+g\|.$$

To prove our Proposition 2, we shall need in section 6 another version of the disjointness lemma, in which the index set is itself a set of ordinals. If A is a set of ordinals, it is itself a well ordered set, and thus order-equivalent to some ordinal. This ordinal is called the order type of A. (Note that it is usually different from the type of A, $\tau(A)$!)

LEMMA 4.1. *Let T be a bounded linear operator on $C(\alpha)$. Assume that $(X_\beta)_{\beta\in\mathcal{B}}$ and $\{A_\beta\}_{\beta\in\mathcal{B}}$ satisfy* (a) *and* (b) *of Proposition* 1, *where $\mathcal{B}$ is a set of ordinals, whose order type is ω^η for some ordinal η. Then for each $\varepsilon > 0$, there is a subset $\mathcal{B}_1$ of $\mathcal{B}$, of the same order type, and sets $\{B_\beta\}_{\beta\in\mathcal{B}_1}$ satisfying* (1) *and* (2) *of Proposition* 1.

PROOF. If ω^η is a cardinal, this is exactly Proposition 1. (ω^η is a cardinal iff η is an uncountable cardinal, or $\eta = 1$.) If not, we shall use the same idea as in the proof of Proposition 1 and induction to show how to reduce ε to $3\varepsilon/4$ in condition (2). Indeed, let λ be the cofinality of ω^η. We can find a non-decreasing net $\{\eta_\gamma: \gamma < \lambda\}$ with $\eta_\gamma < \eta$ for all γ, and an increasing net $\{\beta_\gamma: \gamma < \lambda\} \subset \mathscr{B}$, cofinal in $\mathscr{B}$, such that for each $\gamma < \lambda$ the set $\mathscr{B}^\gamma = \{\beta \in \mathscr{B}: \beta_\gamma < \beta \leqq \beta_{\gamma+1}\}$ has order type ω^{η_γ}. By the inductive hypothesis we can assume that for each γ, $(X_\beta, A_\beta)_{\beta\in\mathscr{B}^\gamma}$ satisfy conditions (1) and (2) of Proposition 1 for $\varepsilon/4$. We now re-index the family $\{\mathscr{B}^\gamma: \gamma < \lambda\}$ as $\{\mathscr{B}(\delta, \gamma): \delta, \gamma < \lambda\}$, and show that one of the following holds:

Case I. There is a $\bar{\delta}$ such that for each $\gamma < \lambda$ there is a subset $\mathscr{B}_1^\gamma$ of $\mathscr{B}(\bar{\delta}, \gamma)$ of the same order type, and for each $\beta \in \mathscr{B}_1^\gamma$ we have $\tau(B_\beta) = \tau(A_\beta)$. Here $B_\beta = \{ a \in A_\beta : \sup | Tf(a) | \leqq \frac{1}{2}\varepsilon \|f\|\}$ where the sup is taken over all $f \in \mathrm{sp}\{X_\alpha : \alpha \in \mathscr{B}(\bar{\delta}, \gamma'), \gamma' \neq \gamma, \gamma' < \lambda\}$. We then take $\mathscr{B}_1 = \bigcup \{\mathscr{B}_1^\gamma: \gamma < \lambda\}$, and condition (2) will hold for these $\mathscr{B}_1$ and B_β's and for $3\varepsilon/4$.

Case II. If the above does not hold, then for each $\delta < \lambda$ there is a $\gamma(\delta) < \lambda$ and a subset $\mathscr{B}_1^\delta$ of $\mathscr{B}(\delta, \gamma(\delta))$ of the same order type, such that for each $\beta \in \mathscr{B}_1^\delta$, $\tau(B_\beta) = \tau(A_\beta)$. Here $B_\beta = \{a \in A_\beta : \sup | Tf(a) | \leqq \frac{1}{2}\varepsilon \|f\|\}$, where the sup is taken over all $f \in \mathrm{sp}\{X_\alpha: \delta' < \lambda, \delta' \neq \delta, \alpha \in \mathscr{B}(\delta', \gamma(\delta'))\}$.

We now take $\mathscr{B}_1 = \bigcup \{\mathscr{B}_1^\delta: \delta < \lambda\}$ and condition (2) holds for these $\mathscr{B}_1$ and B_β's and for $3\varepsilon/4$.

In either case $\mathscr{B}_1$ has order type ω^η.

The details are left to the reader.

§5. Proof of Proposition 2

In this section we shall first prove Proposition 2 for uncountable regular cardinals. We then formulate Proposition 3 and proceed to prove Proposition 2 for successor ordinals. Proposition 3 is really the heart of the whole construction and its proof will be given in sections six and seven.

The first lemma is very simple and well known.

LEMMA 5.1. (a) *Let η be an ordinal with uncountable cofinality. Then every continuous function f on $[1, \eta)$ is eventually constant, i.e. there is a $\xi < \eta$ such that f is constant on $[\xi, \eta)$.*

(b) *Let $\{f_\nu\}_{\nu<\lambda}$ be a pointwise convergent net of continuous functions on $[1, \eta)$, where η has uncountable cofinality and λ is a regular uncountable cardinal, and let $f = \lim f_\nu$, then f is eventually constant.*

PROOF. (a) If this were not the case, we would be able to find two increasing uncountable nets $\alpha_\rho < \beta_\rho < \alpha_{\rho+1}$ such that $f(\beta_\rho) > f(\alpha_\rho)$ for all ρ. But then we would be able to find an $\varepsilon > 0$ and two sequences $\alpha_1 < \beta_1 < \alpha_2 < \cdots$ such that $|f(\beta_n) - f(\alpha_n)| \geqq \varepsilon$ for all n, contradicting the continuity of f at $\alpha = \lim \alpha_n = \lim \beta_n$.

(b) If this were not the case we would be able to find $\varepsilon > 0$ and $\alpha_1 < \beta_1 < \cdots$ as above. Since $f_\nu(\alpha_n) \to f(\alpha_n)$ and $f_\nu(\beta_n) \to f(\beta_n)$, and λ is regular and uncountable, there is a ν_0 such that $f_{\nu_0}(\alpha_n) = f(\alpha_n)$, $f_{\nu_0}(\beta_n) = f(\beta_n)$ for all n, contradicting the continuity of f_{ν_0}.

PROOF OF PROPOSITION 2 FOR UNCOUNTABLE REGULAR α. Note first that in this case $\omega^{(\omega^\alpha)} = \alpha$. For $\nu < \alpha$, let χ_ν be the characteristic function of $[0, \nu]$ and $f_\nu = T\chi_\nu$. Since $\{\chi_\nu\}$ is a weak Cauchy net in $C_0(\alpha)$, $\{f_\nu\}$ converges pointwise to some function f on $[0, \alpha]$. Since $f_\nu(\alpha) = 0$ for all ν, we get by Lemma 5.1(a) that each f_ν is eventually zero. By Lemma 5.1(b) there is a $\beta_0 < \alpha$ and a constant c such that $f(\beta) = c$ for all $\beta_0 \leqq \beta < \alpha$. We now choose for each $\rho < \alpha$, $a_\rho < \alpha$ and $\nu(\rho) < \alpha$ such that $a_\rho < \nu(\rho) < a_{\rho+1}$ for all ρ and such that

$$f_{\nu(\rho)}(a_\tau) = \begin{cases} c & \tau \leqq \rho \\ 0 & \tau > \rho. \end{cases}$$

Indeed, let $a_1 = \beta_0$. Then $f(a_1) = c$ and by the regularity of α there is a $\nu(1)$ such that $f_\nu(a_1) = c$ for all $\nu \geqq \nu(1)$.

Inductively assume that a_ρ, $\nu(\rho)$ were chosen for all $\rho < \rho_0$. Each $f_{\nu(\rho)}$ is eventually zero and $\rho_0 < \alpha$, hence by the regularity of α, there is a $a_{\rho_0} > \sup\{\nu(\rho): \rho < \rho_0\}$ such that $f_{\nu(\rho)}(a_{\rho_0}) = 0$ for all $\rho < \rho_0$. Using the regularity of α again, and the fact that $f(a_{\rho_0}) = c$ there is a $\nu(\rho_0) > a_{\rho_0}$ such that $f_\nu(a_{\rho_0}) = c$ for all $\nu \geqq \nu(\rho_0)$.

Let now $A_0 = \{a_\rho : \rho < \alpha \text{ is a successor}\}$ and $A = \bar{A}_0$. The set A is homeomorphic to $[1, \alpha]$ and the space $X = \overline{\mathrm{sp}}\{\chi_{\nu(\rho)}: \rho < \alpha\}$ is determined over A with $R_A(X) = C_0(A)$. Moreover, if $a \in A_0$ and $f = \Sigma_1^n b_j \chi_{\nu(\rho_j)}$, then $f(a) = \Sigma b_j$ where the sum is taken over all j with $\nu(\rho_j) \geqq a$. Similarly $Tf(a) = (\Sigma b_j f_{\nu(\rho_j)})(a) = c \Sigma b_j$, and the sum is taken over the same j's as above. Thus $Tf(a) = cf(a)$ and X is $(c, 0)$ preserved by T over A.

In order to prove the second case of Proposition 2, we shall need the following proposition whose proof will be given in the next two sections.

PROPOSITION 3. *For every ordinal η, and numbers $n, \delta, \rho > 0$, there is an $m = m(\eta, n, \rho, \delta)$ such that for every bounded linear operator T on $C_0(\omega^{\eta \cdot m})$ with $\|T\| \leqq \rho$, there is a closed subset A_n of $[1, \omega^{\eta \cdot m}]$, homeomorphic to $[1, \omega^{\eta \cdot n}]$, a* *number c and a closed subspace X_n of $C_0(\omega^{\eta\cdot m})$ which is determined over A_n, $R_{A_n}(X_n) = C_0(A_n)$ and is (c, δ)-preserved by T over A_n.*

PROOF OF PROPOSITION 2 FOR SUCCESSOR α. Assume that $\alpha = \alpha_1 + 1$, and use Proposition 3 above for $\eta = \omega^{\alpha_1}$, to find, for each n, numbers m_n corresponding to $\rho = \|T\|$, $\delta = \varepsilon/4$ and n.

Notice that $\omega^{(\omega^\alpha)} = \omega^{\eta\cdot\omega}$, and thus we can find disjoint clopen intervals E_n in $[1, \omega^{\eta\cdot\omega}]$ homeomorphic to $[1, \omega^{\eta\cdot m_n}]$. Let $Z_n = \{f \in C_0(\omega^{\eta\cdot\omega}) : f(e) = 0$ for all $e \notin E_n\}$. By Proposition 1 there is an infinite subset M of the integers, and for each $n \in M$ a subset D_n of E_n homeomorphic to E_n such that $|Tf(d)| \leqq \frac{1}{2}\varepsilon \|f\|$ for all $n_0 \in M$, $d \in D_{n_0}$ and $f \in \operatorname{sp}\{Z_n : n \in M, n \neq n_0\}$. Let $S_n : C(D_n) \to C(E_n)$ be the simultaneous extension operator from Lemma 1.1(c), and consider S_n as an operator into $C_0(\omega^{\eta\cdot\omega})$ by defining Sf to be zero outside E_n. We now use Proposition 3 for the operator $T_n = R_{D_n}TS_n : C(D_n) \to C(D_n)$ to get a set $A_n \subset D_n$ homeomorphic to $[1, \omega^{\eta\cdot n}]$, a number c_n and X_n as in that proposition. We can now find an infinite subset M_1 of M and a number c such that $|c_n - c| < \varepsilon/4$ for all $n \in M_1$, and thus X_n is $(c, \varepsilon/2)$ preserved by T_n over A_n. For each $n \in M_1$ choose a subset $A'_n \subset A_n$, homeomorphic to $[1, \omega^{\eta(n-1)}]$ with $\sup A'_n < \sup A_n$. By Lemma 1.2(a) there is a subspace Y_n of X_n, determined over A'_n with $R_{A'_n}(Y_n) = C(A'_n)$. The set $A = \overline{\bigcup\{A'_n : n \in M_1\}}$, the space $X = \overline{\operatorname{sp}}\{Y_n : n \in M_1\}$ and this c satisfy Proposition 2.

REMARKS. (1) Notice that we used Proposition 3 for η of the form $\eta = \omega^\alpha$ only. The proof also reduces easily to this case. Indeed, given η, choose α and k such that $\omega^\alpha \leqq \eta \leqq \omega^\alpha \cdot k$. If Proposition 3 holds for ω^α and n is given, choose $m = m(\omega^\alpha, nk, \rho, \delta)$. Since $[1, \omega^{\eta\cdot m}]$ contains an initial segment homeomorphic to $[1, \omega^{(\omega^\alpha\cdot m)}]$, and $[1, \omega^{(\omega^\alpha\cdot nk)}]$ contains an initial segment homeomorphic to $[1, \omega^{\eta\cdot n}]$, Proposition 3 follows for η and n with this m.

(2) As we saw, Proposition 3 for $\eta = \omega^\alpha$ implies that Proposition 2 holds for $\alpha + 1$. If α is a successor or an uncountable regular cardinal, we will assume in the proof of Proposition 3 for ω^α that Proposition 2 holds for this α. This gives an inductive process that proves both Proposition 2 and 3, except that if α is a limit ordinal (which is not an uncountable regular cardinal) we shall need a somewhat more complicated procedure to prove Proposition 3 for ω^α from the inductive hypothesis that Proposition 2 holds for every successor ordinal $\beta < \alpha$.

§6. Proof of Proposition 3

The proof of Proposition 3 involves two ingredients. The first is a canonical decomposition, for any ordinal γ, of $C_0(\gamma^n)$ into subspaces isometric to $C_0(\gamma)$, and the study of an operator T on $C_0(\gamma^n)$ in terms of its behaviour on these smaller spaces. This will be done in the first part of this section.

The second ingredient is the application of Proposition 2. Here we shall use the above decomposition for $\gamma = \omega^{(\omega^\alpha)}$ and use it to study the operator T by knowing (assuming Proposition 2 for α) that T behaves "nicely" on the smaller spaces isometric to $C_0(\omega^{(\omega^\alpha)})$.

To this end we shall need, however, a stronger form of Proposition 2 (Lemma 6.3). The last section will be devoted to the technical part of showing that Proposition 2 indeed implies this stronger form.

Before we give the description of $C_0(\gamma^n)$, which requires a somewhat messy notation, we start with the case $n = 2$. Each $a < \gamma^2$ has a unique representation as $a = \gamma a_1 + a_2$ with $0 \leqq a_i < \gamma$, and we denote this a by $a = (a_1, a_2)$. For each $b_1 < \gamma$ let $A_{b_1} = \{a : \gamma \cdot b_1 < a \leqq \gamma(b_1 + 1)\}$. Each A_{b_1} is a closed interval in $[1, \gamma^2]$ homeomorphic to $[1, \gamma]$. Also set $A = [1, \gamma^2]$. Let $X = \{f \in C_0(\gamma^2)$: f is constant on each $A_{b_1}\}$, and for each b_1 let $X_{b_1} = \{f \in C_0(\gamma^2)$: f is supported in A_{b_1} and $f|_{A_{b_1}} \in C_0(A_{b_1})\}$. The space X and each of the spaces X_{b_1} are isometric to $C_0(\gamma)$, and $C_0(\gamma^2) = \overline{\mathrm{sp}}\{X, \bigcup_{b_1<\gamma} X_{b_1}\}$. Moreover, given a point $a = (a_1, a_2) < \gamma^2$, every $f \in C_0(\gamma^2)$ has a unique representation $f = g + h$ where $g \in \mathrm{sp}\{X, X_{a_1}\}$ and $h \in \overline{\mathrm{sp}}\{X_{b_1} : b_1 \neq a_1\}$. In this representation $\|g\|, \|h\| \leqq 2\|f\|$. This is a very convenient decomposition since $h_1(d) = 0$ for every $d \in A_{a_1}$ and $h_1 \in \overline{\mathrm{sp}}\{X_{b_1} : b_1 \neq a_1\}$.

We now pass to the description of $C_0(\gamma^n)$ for general n. Each $a < \gamma^n$ has a unique representation $a = \gamma^{n-1}a_1 + \cdots + \gamma a_{n-1} + a_n$, where $0 \leqq a_j < \gamma$, and we shall denote this a by $a = (a_1, \cdots, a_n)$. For each $0 \leqq m < n$ and fixed $0 \leqq b_1, \cdots, b_m < \gamma$, let

$$\begin{aligned} A_{(b_1,\cdots,b_m)} = \{a : (b_1, \cdots, b_m, 0, \cdots, 0) < a \\ \leqq (b_1, \cdots, b_m + 1, 0, \cdots, 0)\} \end{aligned}$$

(if $m = 0$ we put $A = [1, \gamma^n]$). Each $A_{(b_1,\cdots,b_m)}$ is a closed interval homeomorphic to $[1, \gamma^{n-m}]$.

Let $X = \{f \in C_0(\gamma^n)$: f is constant on each $A_{b_1}\}$, and similarly, for each fixed $(b_1, \cdots, b_m)$ let $X_{(b_1,\cdots,b_m)}$ be the space of all functions $f \in C_0(\gamma^n)$ such that

(1) f is supported in $A_{(b_1,\cdots,b_m)}$ and vanishes at its end point

$$(b_1, \cdots, b_m + 1, 0, \cdots, 0).$$

(2) For each $0 \leqq b_{m+1} < \gamma$, f is constant on $A_{(b_1,\cdots,b_{m+1})}$.

Again the space X and each of the spaces $X_{(b_1,\cdots,b_m)}$ are isometric to $C_0(\gamma)$, and

$C_0(\gamma^n)=\overline{\mathrm{sp}}\{X_{(b_1,\cdots,b_m)}\colon 0\leqq m<n,\ 0\leqq b_j<\gamma\}$. Moreover, given a point $a=(a_1,\cdots,a_n)<\gamma^n$, every $f\in C_0(\gamma^n)$ has a unique representation $f=g_0+g_1+\cdots+g_{n-1}+h$ where $g_0\in X$, $g_j\in X_{(a_1,\cdots,a_j)}$ for $1\leqq j\leqq n-1$, and $h\in\overline{\mathrm{sp}}\{X_{(b_1,\cdots,b_m)}\colon 1\leqq m\leqq n-1,\ (b_1,\cdots,b_m)\neq(a_1,\cdots,a_m)\}$. In this representation $\|g_k\|,\|h\|\leqq 2\|f\|$ for every $0\leqq k\leqq n-1$. (Indeed $\|g_0+\cdots+g_k\|\leqq\|f\|$ because every value of this function is also a value of f; $g_0+\cdots+g_k$ is constant on each $A_{(b_1,\cdots,b_{k+1})}$ and if j is the last index such that $b_j=a_j$ its value there is $f(a_1,\cdots,a_j,b_{j+1}+1,0,\cdots)$.)

This decomposition is very important. It allows us to compute the values of a function at a point by computing the values of the components, and this is done with good norm estimates on the components (see e.g. the proof of Lemma 6.2(b)).

REMARK. In the description of $C_0(\gamma^n)$ above, instead of taking *all* the spaces $X_{(b_1,\cdots,b_m)}$, we could take a set of indices $\mathcal{B}$ with the following property:

$(*)$ … The zero tuple is in $\mathcal{B}$, and for each $m<n-1$ and $(b_1,\cdots,b_m)\in\mathcal{B}$, $(b_1,\cdots,b_{m+1})\in\mathcal{B}$ for a set of b_{m+1}'s whose order type is γ. (Recall that the order type of a set of ordinals H is the ordinal to which H is order-equivalent.)

In this case there is a subset D of $[1,\gamma^n]$, homeomorphic to $[1,\gamma^n]$, and a subspace Y of $\overline{\mathrm{sp}}\{X_{(b_1,\cdots,b_m)}\colon(b_1,\cdots,b_m)\in\mathcal{B}\}$ such that $R_D\colon Y\to C_0(D)$ is an isometry onto. Indeed, let $D_0=\bigcup\{A_{(b_1,\cdots,b_{n-1})}\colon(b_1,\cdots,b_{n-1})\in\mathcal{B}\}$ and $D=\bar{D}_0$. For each $(b_1,\cdots,b_m)\in\mathcal{B}$ by an argument similar to Lemma 1.2(a) (see also Lemma 6.2(a)), applied to $X_{(b_1,\cdots,b_m)}$, there is a subspace $Y_{(b_1,\cdots,b_m)}$ of $X_{(b_1,\cdots,b_m)}$ such that R_D is an isometry of $Y_{(b_1,\cdots,b_m)}$ onto $R_DX_{(b_1,\cdots,b_m)}$, and we take $Y=\overline{\mathrm{sp}}\{Y_{(b_1,\cdots,b_m)}\colon(b_1,\cdots,b_m)\in\mathcal{B}\}$.

The building blocks in the above description of $C_0(\gamma^n)$ are the spaces $X_{(b_1,\cdots,b_m)}$ which are isometric to $C_0(\gamma)$ and consist of functions which are constant on "large" sets. Supported in these large sets we then construct more copies of $C_0(\gamma)$, namely, the spaces $X_{(b_1,\cdots,b_m,b_{m+1})}$, etc. This is the kind of procedure that we shall follow in constructing $C_0(\omega^{\eta\cdot n})$ in the proof of Proposition 3. To this end we shall need conditions on the operator T to preserve this kind of behaviour. Recall that by Remark (1) at the end of section five, we are interested only in ordinals of the type $\gamma=\omega^{(\omega^\alpha)}$. The following definition and lemmas, however, make sense (and save messy indexing) for ordinals γ of the form $\gamma=\omega^\tau$ for any τ.

We first introduce some notation. Let $A\subset B$ be closed subsets of $[1,\eta]$ for some η. We say that the pair (A,B) *is of type* (α,β) if B is homeomorphic to $[1,\omega^{\beta+\alpha}]$ and $A=B^{(\beta)}$ (in particular, A is homeomorphic to $[1,\omega^\alpha]$).

For each $a \in A$, $a \neq \sup A$, let $a^+ = \inf\{c \in A : c > a\}$. We say that a subspace X of $C_0(\eta)$ is *determined over* (A, B) if it is determined over B, and each function $f \in X$ is constant on each interval $B \cap [a+1, a^+]$, $a \in A$. (Note that in this case X is already determined over A.)

If T is a bounded linear operator on $C_0(\eta)$, we say that X *is* (c, ε)*-preserved by* T *over* (A, B), if it is (c, ε)-preserved by T over B. (This terminology will be used only when X is determined over (A, B) as well.)

If (A, B) and (C, D) are two pairs we say that (C, D) *is contained in* (A, B), denoted by $(C, D) < (A, B)$, if there is an $a \in A$ such that $D \subset [a+1, a^+] \cap B$.

The first lemma will summarize the description of $C_0(\omega^{\alpha \cdot n})$ that we gave in the beginning of this section in terms of subspaces isometric to $C_0(\omega^\alpha)$ and determined over appropriate pairs. The second lemma gives conditions on the behaviour of the operator T on each of these smaller spaces so that the copy of $C(\omega^{\alpha \cdot n})$ will be (c, ε)-preserved by T over some set D homeomorphic to $[1, \omega^{\alpha \cdot n}]$.

In the following two lemmas η will be a fixed ordinal with $\omega^{\alpha \cdot n} \leqq \eta$, and all sets will be closed subsets of $[1, \eta]$.

Lemma 6.1. *Let $\mathcal{B}$ be a set of indices satisfying $(*)$ for $\gamma = \omega^\alpha$ and some α. For each $\bar{b} = (b_1, \cdots, b_m) \in \mathcal{B}$, let $(A_{\bar{b}}, B_{\bar{b}})$ be a pair and $X_{\bar{b}}$ a subspace of $C_0(\eta)$ such that*

1. *If $\bar{b} = (b_1, \cdots, b_m)$ (i.e. has length m), then $(A_{\bar{b}}, B_{\bar{b}})$ has type $(\alpha, \alpha(n-m-1))$.*
2. *If $\bar{b} = (b_1, \cdots, b_{m+1}) \in \mathcal{B}$ and if $\bar{b}_0 = (b_1, \cdots, b_m)$, then $(A_{\bar{b}}, B_{\bar{b}}) < (A_{\bar{b}_0}, B_{\bar{b}_0})$. Also $B_{\bar{b}} \cap B_{\bar{b}'} = \emptyset$ if $\overline{b'} = (b'_1, \ldots, b'_{m+1}) \neq \bar{b}$.*
3. *$X_{\bar{b}}$ is determined over $(A_{\bar{b}}, B_{\bar{b}})$ and $R_{A_{\bar{b}}}(X_{\bar{b}}) = C_0(A_{\bar{b}})$.*
4. *Every function $f \in X_{\bar{b}}$ is supported in $[\xi_1, \xi_2]$ where $\xi_1 = \inf\{\xi : \xi \in B_{\bar{b}}\}$, $\xi_2 = \sup\{\xi : \xi \in B_{\bar{b}}\}$. (Notice that if $f \in X_{(b_1, \cdots, b_{m+1})}$, $g \in X_{(b_1, \cdots, b_m)}$, then g is constant on the support of f.)*

Let $D_0 = \bigcup \{A_{(b_1, \cdots, b_{n-1})} : (b_1, \cdots, b_{n-1}) \in \mathcal{B}\}$, and $D = \bar{D}_0$. Then D is homeomorphic to $[1, (\omega^\alpha)^n]$, and if we put $Z = \overline{\mathrm{sp}}\{X_{\bar{b}} : b \in \mathcal{B}\}$, there is a subspace Y of Z such that Y is determined over D and $R_D(Y) = C_0(D)$ (in particular Y is isometric to $C_0(\omega^{\alpha \cdot n})$).

Proof. The conditions are the same as those given in the description of $C_0(\gamma^n)$ above for $\gamma = \omega^\alpha$, and in the Remark thereafter.

The next lemma is the analogue of Lemma 1.2 for pairs:

Lemma 6.2. (a) *Let X be determined over a pair (A, B) of type (α, β), and D a closed subset of A homeomorphic to $[1, \omega^{\alpha_1}]$ for some $\alpha_1 \leqq \alpha$. If $R_A(X) = C_0(A)$,* *there is a set E, $D \subset E \subset B$ and a subspace Y of X such that (D, E) is a pair of type (α_1, β), Y is determined over (D, E) and $R_D(Y) = C_0(D)$.*

(b) *Let $\mathcal{B}$, $(A_{\mathfrak{b}}, B_{\mathfrak{b}})$, $X_{\mathfrak{b}}$, D and Y be as in Lemma 6.1, and let T be a bounded linear operator on $C_0(\eta)$, $\varepsilon > 0$ and c a number such that:*

(1) *Each $X_{\mathfrak{b}}$ is $(c, \varepsilon/4n)$-preserved by T over $(A_{\mathfrak{b}}, B_{\mathfrak{b}})$.*

(2) *For each $(a_1, \cdots, a_{n-1}) \in \mathcal{B}$, $d \in A_{(a_1, \cdots, a_{n-1})}$ and $h \in \overline{\mathrm{sp}} \bigcup_{m=1}^{n-1} \{X_{(b_1, \cdots, b_m)}$: $(b_1, \cdots, b_m) \in \mathcal{B}$, $(b_1, \cdots, b_m) \neq (a_1, \cdots, a_m)\}$, $|Th(d)| \leqq \frac{1}{4}\varepsilon \| h \|$.*

Then Y is (c, ε)-preserved by T over D. In particular, if $c > 3\varepsilon > 0$, T is an isomorphism of Y onto TY, and TY is complemented in $C_0(\eta)$.

PROOF. Part (a) is proved as Lemma 1.2(a). Notice that the extension operator constructed in Lemma 1.1(c) extends the functions to be constant on the intervals between consecutive points of D, hence every choice of $E \subset B$ such that $E^{(\beta)} = D$ will do.

To prove (b), let $d \in D_0$ be arbitrary and let $f \in Y$. Since the sets $A_{(b_1, \cdots, b_{n-1})}$ are disjoint, there is a unique $(a_1, \cdots, a_{n-1}) \in \mathcal{B}$ such that $d \in A_{(a_1, \cdots, a_{n-1})}$. Let $f = g + h$ be the unique representation with $g \in \mathrm{sp}\{X, X_{(a_1)}, \cdots, X_{(a_1, \cdots, a_{n-1})}\}$, $h \in \overline{\mathrm{sp}}\{X_{(b_1, \cdots, b_m)}$: $1 \leqq m \leqq n-1$ and $(b_1, \cdots, b_m) \neq (a_1, \cdots, a_m)\}$. Also decompose $g = g_0 + g_1 + \cdots + g_{n-1}$ where $g_0 \in X, \cdots, g_{n-1} \in X_{(a_1, \cdots, a_{n-1})}$. Then $\| h \|, \| g_0 \|, \cdots, \| g_{n-1} \| \leqq 2 \| f \|$, and thus

$$| Tf(d) - cf(d) | \leqq \sum_{i=0}^{n-1} | Tg_i(d) - cg_i(d) | + | Th(d) - ch(d) | \leqq \varepsilon \| f \|,$$

since by (1) $| Tg_i(d) - cg_i(d) | \leqq (\varepsilon/4n) \| g_i \| \leqq (\varepsilon/2n) \| f \|$ for all i and by (2) $| Th(d) | \leqq \frac{1}{4}\varepsilon \| h \| \leqq \frac{1}{2}\varepsilon \| f \|$, while $h(d) = 0$ by the definition of h.

The last assertion follows from Lemma 1.2(b) and (c).

We are now ready to formulate the strengthened form of Proposition 2 that we shall need.

LEMMA 6.3. *Let α be an ordinal such that the conclusion of Proposition 2 holds for α. Let γ, β be ordinals with $\gamma \geqq \omega^{\beta + \omega^\alpha}$ and T a bounded linear operator on $C_0(\gamma)$. Let $\xi_1 < \xi_2 \leqq \gamma$ and $C \subset [\xi_1 + 1, \xi_2]$ be a closed set homeomorphic to $[1, \omega^{\beta + \omega^\alpha}]$. Then for every $\varepsilon > 0$, there is a closed subspace Y of $C_0(\gamma)$, a pair (A, B) of type (ω^α, β) with $B \subset C$, and a number c, such that*

(1) *the functions in Y are supported in $[\xi_1 + 1, \xi_2]$,*

(2) *Y is determined over (A, B) and $R_A(Y) = C_0(A)$,*

(3) *Y is (c, ε)-preserved by T over (A, B).*

Notice that Lemma 6.3 differs from Proposition 2 in two ways. The first is that we deal with an operator on $C_0(\gamma)$ instead of $C_0(\omega^{(\omega^\alpha)})$, and we find our "preserved" space and the set on which it is preserved inside given space and set. This difference is just formal and very easy to overcome. The main difference is that the space is not only (c, ε)-preserved over a set homeomorphic to $[1, \omega^{(\omega^\alpha)}]$, but over a *pair* (A, B) of type (ω^α, β), and β is arbitrary. This means that both the space we construct and its image consist of functions that are constant on "large" sets, allowing us to iterate applications of this lemma and construct further spaces of functions supported in these sets, thus giving a structure like in Lemmas 6.1 and 6.2.

PROOF OF PROPOSITION 3 (for $\eta = \omega^\alpha$, α satisfying Proposition 2). Let $\tau < \delta/12n$ and let $m > 2\rho(n+1)/\tau$. The proof will consist of three steps. In step I, which is the main step, we shall construct spaces $X_{\bar a}$, and pairs $(A_{\bar a}, B_{\bar a})$ as in Lemma 6.1, with m "levels," i.e. Y will be isometric to $C_0(\omega^{\eta \cdot m})$. Also for each $\bar a$ there will be a (possibly different) constant $c(\bar a)$ such that $X_{\bar a}$ will be $(c(\bar a), \tau)$-preserved by T over $(A_{\bar a}, B_{\bar a})$. The disjointness condition ((2) in Lemma 6.2(b)) will be also satisfied.

In step II we shall pass to a smaller set of indices $\mathscr{C}$ still satisfying the condition $(*)$ and find numbers $c_0, \cdots, c_{m-1}$ such that if $\bar a \in \mathscr{C}$ has length j, (i.e. $\bar a = (a_1, \cdots, a_j)$) then $X_{\bar a}$ will be $(c_j, 2\tau)$-preserved by T over $(A_{\bar a}, B_{\bar a})$.

In step III we shall use the fact that we have m numbers $c_0, \cdots, c_{m-1}$ in $[-\rho, \rho]$, and find n of them which are essentially the same number c. It is then a simple matter to find a subset $\mathscr{B}$ of C which will be of order type $\omega^{\eta \cdot n}$ for which all the conditions of Lemma 6.2(b) will be satisfied.

Step I. We shall first use Lemma 6.3 with $\gamma = \omega^{\eta \cdot m}$, $\beta = \eta(m-1) = \omega^\alpha(m-1)$, $C = [1, \omega^{\eta \cdot m}]$ and τ for ε. We thus find a pair (A, B) of type $(\eta, \eta(m-1))$, a number c_0 and a subspace X of $C_0(\omega^{\eta \cdot m})$, which is determined over (A, B), $R_A(X) = C_0(A)$, and is (c_0, τ)-preserved by T over (A, B). For each $a_1 \in A$, let $a_1^+ = \inf\{a \in A : a > a_1\}$ and let $Z_{a_1} = \{f \in C_0(\omega^{\eta \cdot m}) : f(e) = 0$ for $e \notin [a_1 + 1, a_1^+)\}$. We would like now to use Lemma 6.3 for $C = B \cap [a_1 + 1, a_1^+]$, however we first want to make sure that the disjointness condition ((2) in Lemma 6.2(b)) will be satisfied. Hence we use Lemma 4.1 to find a subset $\mathscr{A}$ of A, of the same order type ω^η, and for each $a_1 \in \mathscr{A}$ a subset C_{a_1} of $B \cap [a_1 + 1, a_1^+]$ homeomorphic to $\omega^{\eta(m-1)}$ such that $|Tf(d)| \leqq \tau \|f\|$ for all $b_1 \in \mathscr{A}$, $d \in C_{b_1}$ and $f \in \mathrm{sp}\{Z_{a_1} : a_1 \in \mathscr{A}, a_1 \neq b_1\}$. For each $a_1 \in \mathscr{A}$ we now use Lemma 6.3 again with $\beta = \eta(m-2)$, $C = C_{a_1}$, $\xi_1 = a_1$, $\xi_2 = a_1^+$ to find a pair (A_{a_1}, B_{a_1}) of type $(\eta, \eta(m-2))$ with $B_{a_1} \subset C_{a_1}$, and a subspace X_{a_1} supported in $[a_1 + 1, a_1^+]$, determined over (A_{a_1}, B_{a_1}), with $R_{A_{a_1}}(X_{a_1}) = C_0(A_{a_1})$, and which is $(c(a_1), \tau)$-preserved by T over (A_{a_1}, B_{a_1}).

Inductively, given $\bar{a} = (a_1, \cdots, a_j)$ and a pair $(A_{\bar{a}}, B_{\bar{a}})$ of type $(\eta, \eta(m-j-1))$, we define for each $a_{j+1} \in A_{\bar{a}}$, $a_{j+1}^+ = \inf\{a \in A_{\bar{a}}: a > a_{j+1}\}$ and $Z_{a_{j+1}} = \{f \in C(\omega^{\eta \cdot m}): f(e) = 0 \text{ for } e \not\in [a_{j+1}+1, a_{j+1}^+)\}$. We now use Lemma 4.1 to find $\mathcal{A}_{\bar{a}} \subset A_{\bar{a}}$ of the same order type, ω^η, and for each $a_{j+1} \in \mathcal{A}_{\bar{a}}$ a closed subset $C_{a_{j+1}}$ of $[a_{j+1}+1, a_{j+1}^+] \cap B_{\bar{a}}$ homeomorphic to $\omega^{\eta(m-j-1)}$ such that $|Tf(d)| \leqq \tau \|f\|$ for all $b_{j+1} \in \mathcal{A}_{\bar{a}}$, $d \in C_{b_{j+1}}$ and $f \in \operatorname{sp}\{Z_{a_{j+1}}: a_{j+1} \in \mathcal{A}_{\bar{a}}, a_{j+1} \neq b_{j+1}\}$.

We now use Lemma 6.3 again for each $a_{j+1} \in \mathcal{A}_{\bar{a}}$ with $C = C_{a_{j+1}}$, $\beta = \eta(m-j-2)$, $\xi_1 = a_{j+1}$, $\xi_2 = a_{j+1}^+$ and τ to find a pair $(A_{a_1, \cdots, a_{j+1}}, B_{a_1, \cdots, a_{j+1}})$, a space $X_{a_1, \cdots, a_{j+1}}$ and a number $c(a_1, \cdots, a_{j+1})$. Proceeding this way for $j = 1, 2, \cdots, m-1$, we get the required pairs of sets, $(A_{\bar{a}}, B_{\bar{b}})$, and spaces $X_{\bar{a}}$ as in Lemma 6.1.

Step II. c_0 is already given. To find c_1, notice that there is a subset $\mathcal{B}_1$ of $\mathcal{A}$, of the same order type, and c_1 such that $|c_1(b_1) - c_1| < \tau$ for all $b_1 \in \mathcal{B}_1$. This is the c_1 we are looking for.

Next for each $b_1 \in \mathcal{B}_1$ we can find a constant $c(b_1)$ and a subset $\mathcal{B}_{b_1} \subset \mathcal{A}_{b_1}$ of the same order type with $|c(b_1, a_2) - c(b_1)| < \tau/2$ for all $a_2 \in \mathcal{B}_{b_1}$, and then find a subset $\mathcal{C}_1$ of $\mathcal{B}_1$ of the same order type as $\mathcal{B}_1$ and a c_2, such that $|c(b_1) - c_2| < \tau/2$ for all $b_1 \in \mathcal{C}_1$. Iterating this process m-times we find $c_0, \cdots, c_{m-1}$ and $\mathcal{C}$ as required. Thus for each $\bar{b} = (b_1, \cdots, b_j) \in \mathcal{C}$, $X_{\bar{b}}$ is $(c_j, 2\tau)$-preserved by T over $(A_{\bar{b}}, B_{\bar{b}})$.

Step III. Since we have m numbers $c_0, \cdots, c_{m-1}$ all in the interval $[-\|T\|, \|T\|] = [-\rho, \rho]$, and since $m > 2\rho(n+1)/\tau$, we can find n of these numbers $c_{i_1}, \cdots, c_{i_n}$ and a number c such that $|c_{i_k} - c| < \tau$, $k = 1, \cdots, n$. By picking only from the n levels $i_1, \cdots, i_n$, we can pass to a subset $\mathcal{B}$ of $\mathcal{C}$, which satisfies $(*)$ — this time for n and not for m.

Indeed, fix any $(b_1, \cdots, b_{i_1}) \in \mathcal{C}$, and let $X = X_{(b_1, \cdots, b_{i_1})}$. For each b_{i_1+1} such that $(b_1, \cdots, b_{i_1}, b_{i_1+1}) \in \mathcal{C}$ choose $d_1 = (b_1, \cdots, b_{i_1}, b_{i_1+1}, \cdots, b_{i_2}) \in \mathcal{C}$ and let $X_{d_1} = X_{(b_{i_1}, \cdots, b_{i_2})}$, $A_{d_1} = A_{(b_1, \cdots, b_{i_2})}$, $B_{d_1} = B_{(b_1, \cdots, b_{i_2})}$. Next, for each fixed $d_1 = (b_1, \cdots, b_{i_2})$ and each b_{i_2+1} with $(b_1, \cdots, b_{i_2}, b_{i_2+1}) \in \mathcal{C}$, find $(d_1, d_2) = (b_1, \cdots, b_{i_2}, b_{i_2+1}, \cdots, b_{i_3}) \in \mathcal{C}$. Let $X_{(d_1, d_2)} = X_{(b_{i_1}, \cdots, b_{i_3})}$ and define $A_{(d_1, d_2)}$, $B_{(d_1, d_2)}$ similarly.

We repeat this process for the n levels $i_1, \cdots, i_n$. The set $\mathcal{B} = \{\bar{d} = (d_1, \cdots, d_k): 0 \leqq k < n\}$ satisfies $(*)$ (because $\mathcal{C}$ did, and by the choice of the d's).

The pairs $(A_{\bar{d}}, B_{\bar{d}})$ and the spaces $X_{\bar{d}}$ for $d \in \mathcal{B}$ and this c are those which satisfy the conditions of Lemma 6.2(b). Indeed for each $\bar{d} \in \mathcal{B}$, $X_{\bar{d}}$ is $(c, 3\tau)$-preserved by T over $(A_{\bar{d}}, B_{\bar{d}})$ and $3\tau < 3\delta/12n = \delta/4n$. Also the "disjointness condition" (2) is satisfied, since in each level we have τ-disjointness and we have n levels and $n\tau < \delta/12 < \delta/4$.

PROOF OF PROPOSITION 3 (for $\eta = \omega^\alpha$, α a limit ordinal which is not an uncountable regular cardinal). This case is similar to the previous one, and we indicate briefly the necessary modifications. Let λ be the cofinality of α and $\{\beta_\xi : \xi < \lambda\}$ a net increasing to α, such that each β_ξ is a successor. Since α is not a regular cardinal, we can assume that $\beta_\xi \geqq \lambda$ for all $\xi \geqq 1$. To simplify notation, let $\eta_\xi = \omega^{\beta_\xi}$.

This time we also have to prove Proposition 3 for $n = 1$ (which was, with $m = 1$, exactly what Proposition 2 ensured in the previous case). This is what we now indicate.

Consider first an operator T on $C_0(\omega^\eta)$. By using Proposition 2 for each β_ξ, we can find sets $A_\xi \subset [1, \omega^\eta]$, homeomorphic to $[1, \omega^{\eta_\xi}]$ and contained in disjoint intervals, subspaces X_ξ and numbers c_ξ, such that X_ξ is determined and (c_ξ, τ)-preserved by T over A_ξ, and $R_{A_\xi}(X_\xi) = C_0(A_\xi)$. By passing to a subset of the ξ's, of the same cardinality, we can assume that there is a c such that $|c_\xi - c| < \tau$, and also (using Proposition 1) that the X_ξ's satisfy the "disjointness condition" for τ. Thus $X = \overline{\mathrm{sp}}\{X_\xi\}$ is determined and $(c, 3\tau)$-preserved over $A = \overline{\bigcup A_\xi}$ and $R_A(X) = (\Sigma_\xi \oplus C_0(A_\xi))_0$.

To get a space whose restriction to A is $C_0(A)$, we should have in X also functions $\{f_\rho : \rho < \lambda\}$ such that

$$f_\rho(A_\xi) = \begin{cases} 1 & \xi \leqq \rho \\ 0 & \xi > \rho. \end{cases}$$

And these f_ρ's should also be (c, τ)-preserved over A.

To this end we go from ω^η to $\omega^{\eta \cdot 2}$. Write $[1, \omega^{\eta \cdot 2}]$ as a disjoint union of clopen intervals $\{C_\xi : \xi < \lambda\}$, homeomorphic to $\omega^\eta \cdot \omega^{\eta_\xi}$ respectively. On each C_ξ we use Lemma 6.3 with $\beta = \eta$ and $\alpha = \beta_\xi$. We thus find pairs (A_ξ, B_ξ) of type (η_ξ, η), spaces X_ξ and numbers c_ξ, such that X_ξ is determined over (A_ξ, B_ξ) with $R_{A_\xi}(X_\xi) = C_0(A_\xi)$, and X_ξ is (c_ξ, τ)-preserved by T over (A_ξ, B_ξ). By passing to a smaller set of ξ's of the same cardinality, and using the disjointness lemma (Proposition 1), we can also assume that $c_\xi = c_0$ is independent of ξ and that the X_ξ's satisfy the disjointness condition. Thus if we let $A = \overline{\bigcup A_\xi}$, $B = \overline{\bigcup B_\xi}$, then (A, B) is a pair of type (η, η) and $X = \overline{\mathrm{sp}}(X_\xi)$ is determined and (c_0, τ)-preserved by T over (A, B) with $R_A(X) = (\Sigma_\xi \oplus C_0(A_\xi))_0$.

As in the previous case, we now define for each $a \in A$, $a^+ = \inf\{e \in A : e > a\}$, and repeat the construction on each of the sets $[a + 1, a^+] \cap B$, to find for each $\xi < \lambda$ a subset $A(a,\xi)$ homeomorphic to $[1,\omega^{\eta_\xi}]$ and a space $X(a,\xi)$ determined and $(c(a,\xi),\tau)$-preserved over $A(a,\xi)$ by T with $R_{A(a,\xi)}(X_{(a,\xi)}) = C_0(A(a,\xi))$. By passing to a subset of the a's of the same cardinality, and for each of these a's to subsets of the ξ's of the same cardinality, we can assume that $c(a,\xi) = c_1$ is independent of a and ξ, and that the spaces $X(a,\xi)$ satisfy the disjointness condition.

This is the point where the construction is essentially different from the previous case. We are not going to use all the first "level," we shall use it only to find the functions f_ρ. Similarly we shall not use all of the second level, but pick only a suitable collection of the spaces $X(a,\xi)$.

Pick any space X_ξ from the first "level," say X_1. Since $\beta_1 \geqq \lambda$ there is an increasing net $\{a(\rho): \rho < \lambda\}$ in A_1, and let $f_\rho \in X_1$ be the unique function in X_1 such that $f_\rho(a) = 1$ if $a \in A$, $a \leqq a_\rho$ and $f_\rho(a) = 0$ if $a \in A$, $a > a(\rho)$. For each $\rho < \lambda$ pick now the ρth set $A(a(\rho),\rho)$ in the $a(\rho)$th block and the ρth space $X(a(\rho),\rho)$ from the second level. Again $D = \overline{\bigcup_\rho A(a(\rho),\rho)}$ is homeomorphic to ω^η and $X = \overline{\mathrm{sp}}\{X(a(\rho),\rho)\}$ is determined over D with $R_D(X) = (\Sigma_\rho \oplus C_0(A(a(\rho),\rho)))_0$. But this time the functions f_ρ that we have chosen behave exactly as they should:

$$f_\tau(A(a(\rho),\rho)) = \begin{cases} 1 & \tau \geqq \rho \\ 0 & \tau < \rho. \end{cases}$$

Thus $Y = \overline{\mathrm{sp}}[\{f_\rho : \rho < \lambda\} \cup X]$ is determined over D and $R_D(Y) = C_0(D)$. The trouble is that the f_ρ's are (c_0,τ)-preserved by T over D, and X is (c_1,τ)-preserved by T over D, and c_0 might be different from c_1.

But now we argue as in the proof of the previous case that if $T: C_0(\omega^{\eta\cdot m}) \to C_0(\omega^{\eta\cdot m})$, $\|T\| \leqq \rho$ and m is large enough, we can find m "levels" and m numbers $c_0, \cdots, c_{m-1}$ as above (instead of just $m = 2$), and then there are i_1, i_2 such that $|c_{i_1} - c_{i_2}| < \tau$. We then do the above construction using the i_1, i_2 levels instead of the first and second.

This proves the proposition for $n = 1$. The general case is similar.

§7. Proof of Lemma 6.3

As we mentioned after the formulation of Lemma 6.3, its main difference from Proposition 2 is that it ensured the existence of a space which is preserved by T over a *pair* and not only over a set.

This, the passage from a set to a pair, is the content of the first and main lemma in this section.

After formulating Lemma 7.1 we easily deduce Lemma 6.3 from it. We then prove two technical lemmas on pointwise converging nets of continuous functions, and finish the section with a proof of Lemma 7.1.

LEMMA 7.1. *Let η, β, γ be ordinals such that $\gamma \geqq \omega^{\beta+\eta}$, T a bounded linear operator on $C_0(\gamma)$. Let X be a closed subspace of $C_0(\gamma)$, $E \subset D \subset [1, \gamma]$ closed subsets of $[1, \gamma]$, c and $\varepsilon > 0$ be given such that*

(1) (E, D) is a pair of type (η, β).

(2) X is determined over (E, D) and $R_E(X) = C_0(E)$.

(3) X is (c, ε)-preserved by T over E.

Then for every $\varepsilon_1 > \varepsilon$ there is a pair (A, B) and a subspace Y of X such that

(a) (A, B) is a pair of type (η, β) with $B \subset D$, $A \subset E$.

(b) Y is determined over (A, B) and $R_A(Y) = C_0(A)$.

(c) Y is (c, ε_1)-preserved by T over (A, B).

Lemma 6.3 follows from Lemma 7.1. Indeed, the set $F = C^{(\beta)}$ is homeomorphic to $[1, \omega^{(\omega^\alpha)}]$. By Lemma 1.1(c) there is a simultaneous extension operator S: $C(F) \to C(\xi_1 + 1, \xi_2)$, and we can consider S as an operator from $C(F)$ to $C(\gamma)$ by putting $Sf(a) = 0$ if $a \notin [\xi_1 + 1, \xi_2]$. We now consider the operator $T_1 = R_F TS$: $C(F) \to C(F)$. Since Proposition 2 holds for α and since F is homeomorphic to $[1, \omega^{(\omega^\alpha)}]$, we can find a subset E of F, homeomorphic to $[1, \omega^{(\omega^\alpha)}]$, a number c, and a subspace Z of $C(F)$, which is determined over E, $R_E(Z) = C_0(E)$, and which is $(c, \varepsilon/2)$-preserved by T_1 over E. Since $E \subset F \subset C^{(\beta)}$, we can find a $D \subset C$ such that (E, D) is a pair of type (ω^α, β). We now use Lemma 7.1 for the operator T, the space $X = SZ$, $\eta = \omega^\alpha$, $\varepsilon_1 = \varepsilon$ and this pair (E, D) to find (A, B) and Y as required.

The next lemma shows that an uncountable net of continuous functions converging to a continuous function must eventually be equal to the limit on a large set.

LEMMA 7.2. *Let λ be an uncountable regular cardinal, τ any ordinal. Let $\{f_\nu\}_{\nu<\lambda}$ be a net of continuous functions on $[1, \omega^\tau]$, converging pointwise to zero. Then there is a closed subset H of $[1, \omega^\tau]$, homeomorphic to $[1, \omega^\tau]$ and a family $\mathcal{F} \subset \{f_\nu\}_{\nu<\lambda}$ of the same cardinality as λ such that $f(a) = 0$ for all $f \in \mathcal{F}$ and $a \in H$.*

PROOF. Notice that since λ is regular and uncountable, the condition $f_\nu(a) \to 0$ means that there is a $\nu(a)$ such that $f_\nu(a) = 0$ for all $\nu \geqq \nu(a)$. The proof will be by induction on τ. In fact we shall use (and prove) a stronger inductive hypothesis, namely, that for each τ, there is an inductive procedure to choose H and $\mathcal{F}$, which we now describe:

For each $\rho < \lambda$, there is a closed set $H_\rho \subset [1, \omega^\tau]$, and an ordinal $\nu(\rho)$ such that

(1) $H_\rho \cap H_{\rho_1} = \varnothing$, $\nu(\rho) < \nu(\rho_1)$ for all $\rho < \rho_1$.

(2) $f_\nu(a) = 0$ for all $a \in H_\rho$ and $\nu \not\in [\nu(\rho)+1, \nu(\rho+1))$, $\nu \geqq \nu(1)$.

(3) $H = H_1 \cup \overline{\bigcup \{H_{\tau(\rho)}: \rho < \lambda\}}$ is homeomorphic to $[1, \omega^\tau]$, for each increasing net $\{\tau(\rho): \rho < \lambda\}$.

It is clear that this H and $\mathcal{F} = \{f_{\nu(\rho)}: \rho < \lambda\}$ satisfy the requirements.

Observe that the lemma and the above procedure are trivial if $\omega^\tau < \lambda$. We just take $H = H_1 = [1, \omega^\tau]$, ($H_\rho = \varnothing$ if $\rho > 1$), and find $\nu_0 < \lambda$ such that $f_\nu(a) = 0$ for all $\nu \geqq \nu_0$, $a \leqq \omega^\tau$ (such ν_0 exists by the regularity of λ). We then take $\nu(\rho) = \nu_0 + \rho$.

Let δ be the cofinality of ω^τ. If $\delta > \lambda$ the procedure is again trivial. By Lemma 5.1(a) each f_ν is eventually constant, and since $f_\nu(\omega^\tau) \to 0$ there is a $\nu_0 < \lambda$ such that $f_\nu(\omega^\tau) = 0$ for all $\nu \geqq \nu_0$, and thus this constant is zero. Since $\delta > \lambda$ we can find a point $\xi < \omega^\tau$ such that, in fact, $f_\nu(a) = 0$ for *all* $\nu \geqq \nu_0$ and *all* $a \geqq \xi$. We now take $H_1 = [\xi, \omega^\tau]$ ($H_\rho = \varnothing$ if $\rho > 1$) and $\nu(\rho) = \nu_0 + \rho$.

It is these two cases ($\omega^\tau < \lambda$ and $\delta > \lambda$) that account for the special role of H_1 in the procedure that we described.

We have two cases left to consider: $\delta < \lambda$ and $\delta = \lambda$. In these cases we shall use the inductive hypothesis. Thus let $\{\gamma_\xi: \xi < \delta\}$ be a net increasing to ω^τ. We can (and will) assume that for each $\xi < \delta$, the interval $[\gamma_\xi + 1, \gamma_{\xi+1}]$ is homeomorphic to $[1, \omega^{\tau_\xi}]$ for some $\tau_\xi < \tau$, where $\{\tau_\xi: \xi < \delta\}$ is a non-decreasing net. By the inductive hypothesis we can find for each $\xi < \delta$ and $\rho < \lambda$, sets $H^\xi_\rho \subset [\gamma_\xi + 1, \gamma_{\xi+1}]$ and ordinals $\nu(\xi, \rho) < \lambda$ such that

(1) For all ξ and all $\rho < \rho_1$, $H^\xi_\rho \cap H^\xi_{\rho_1} = \varnothing$ and $\nu(\xi, \rho) < \nu(\xi, \rho_1)$.

(2) $f_\nu(a) = 0$ for $a \in H^\xi_\rho$ and $\nu \not\in [\nu(\xi, \rho)+1, \nu(\xi, \rho+1))$, $\nu \geqq \nu(\xi, 1)$.

(3) $H^\xi_1 \cup \overline{\bigcup \{H^\xi_{\tau(\rho)}: \rho < \lambda\}}$ is homeomorphic to $[1, \omega^{\tau_\xi}]$ for each increasing net $\{\tau(\rho): \rho < \lambda\}$.

We shall now combine the selections on each of the intervals $[\gamma_\xi + 1, \gamma_{\xi+1}]$ to one selection for $[1, \omega^\tau]$. This is done by a standard "gliding hump" procedure. The details are a little different in the two cases.

$\underline{\delta < \lambda.}$ We shall find an increasing net $\{\mu(\rho): \rho < \lambda\}$ such that $H_\rho = \overline{\bigcup \{H^\xi_1 : \xi < \delta\}}$. will satisfy the requirements for a suitable choice of $\nu(\rho)$.

We let $\mu(1) = 1$. By the regularity of λ, we can find $\nu(1) > \sup\{\nu(\xi, 2): \xi < \delta\}$ with $\nu(1) < \lambda$. Then clearly $f_\nu(a) = 0$ for all $\nu \geqq (1)$ and $a \in H_1 = \overline{\bigcup \{H^\xi_1 : \xi < \delta\}}$.

Inductively assume that $\{\nu(\rho): \rho \leqq \rho_0\}$ and $\{\mu(\rho): \rho < \rho_0\}$ have already been chosen. By (2), for each $\xi < \delta$ there is an $\eta(\xi)$ such that $f_\nu(a) = 0$ for all $\nu(1) \leqq \nu \leqq \nu(\rho_0)$ and $a \in H^\xi_\mu$, if $\mu \geqq \eta(\xi)$. Since $\delta < \lambda$ we can find $\mu(\rho_0) < \lambda$ such that $\mu(\rho_0) > \sup\{\eta(\xi) : \xi < \delta\} + \sup\{\mu(\rho) : \rho < \rho_0\}$.

We now use (2) and the regularity of λ again to find $\nu(\rho_0 + 1)$ such that if $\nu \geqq \nu(\rho_0 + 1)$ then $f_\nu(a) = 0$ for all $a \in H^\xi_{\mu(\rho_0)}$ for all ξ.

If ρ_1 is a limit ordinal we just take $\nu(\rho_1) = \sup\{\nu(\rho) : \rho < \rho_1\}$.

These $\nu(\rho)$ and $H_\rho = \overline{\bigcup\{H^\xi_{\mu(\rho)} : \xi < \delta\}}$ satisfy the required conditions. Indeed (1) and (2) are obvious, and if $\{\tau(\rho) : \rho < \lambda\}$ is arbitrary then $H_1 \cup \overline{\bigcup H_{\tau(\rho)}} = \overline{\bigcup_\xi [H^\xi_1 \cup (\bigcup H^\xi_{\tau(\rho)})]}$, and this is the closure of a union of disjoint sets homeomorphic to $[1, \omega^{\tau_\xi}]$. By the choice of τ_ξ such a set is homeomorphic to $[1, \omega^\tau]$.

$\delta = \lambda$. We first note that by the same argument as in the case $\delta > \lambda$ (namely, by using Lemma 5.1), there is a $\nu_0 < \lambda$ such that for each $\nu \geqq \nu_0$ there is a $\xi = \xi(\nu)$ such that $f_\nu(a) = 0$ for all $a \geqq \gamma_\xi$. To simplify notation assume that $\nu_0 = 1$.

We now define two increasing nets $\{\mu(\rho) : \rho < \lambda\}$ and $\{\xi(\rho) : \rho < \lambda\}$ such that $H_\rho = H^{\xi(\rho)}_1 \cup \overline{\bigcup\{H^{\xi(\eta)}_{\mu(\rho)} : \eta \leqq \rho\}}$ will satisfy the requirements for suitable $\nu(\rho)$'s.

Let $\nu(1) = \nu(1,1)$, $\mu(1) = \xi(1) = 1$, $\nu(2) = \nu(1,2)$. Inductively assume that $\{\xi(\rho), \mu(\rho) : \rho < \rho_0\}$ and $\{\nu(\rho) : \rho \leqq \rho_0\}$ have already been chosen. By the regularity of λ and the observation in the first paragraph of this case, we can find $\xi(\rho_0) < \lambda$ such that $f_\nu(a) = 0$ for all $\nu \leqq \nu(\rho_0)$ and $a \geqq \gamma_{\xi(\rho_0)}$.

By the same argument as in the case $\delta < \lambda$ we can find $\mu(\rho_0)$ such that $f_\nu(a) = 0$ for all $\nu(1) \leqq \nu \leqq \nu(\rho_0)$ and $a \in H^\xi_\mu$ whenever $\xi \leqq \xi(\rho_0)$ and $\mu \geqq \mu(\rho_0)$. Again by the same argument as before we can find $\nu(\rho_0 + 1)$ such that if $\nu \geqq \nu(\rho_0 + 1)$, $f_\nu(a) = 0$ for all $a \in H_{\rho_0} = H^{\xi(\rho_0)}_1 \cup \overline{\bigcup\{H^{\xi(\eta)}_{\mu(\rho_0)} : \eta \leqq \rho_0\}}$.

If ρ_1 is a limit ordinal we take $\nu(\rho_1) = \sup\{\nu(\rho) : \rho < \rho_1\}$. These $\nu(\rho)$ and H_ρ satisfy the required conditions.

Lemma 7.3 is an easy consequence of the previous lemma. Note that we no longer require the limit to be continuous.

LEMMA 7.3. *Let λ be an uncountable regular cardinal, τ an ordinal of cofinality λ and $\{\tau_\xi : \xi < \lambda\}$ a net increasing to τ. Let $\{h_\nu\}_{\nu<\lambda}$ be a pointwise convergent net of continuous functions on $[1, \omega^\tau]$. Then there are constants c_1 and c_2 (not necessarily distinct), and for each $\xi < \lambda$ a closed set H_ξ and an index $\nu(\xi)$ such that*

(1) *For each $\xi_1 < \xi_2$, H_{ξ_1} and H_{ξ_2} are contained in disjoint intervals and $\nu(\xi_1) < \nu(\xi_2)$.*

(2) *H_ξ is homeomorphic to $[1, \omega^{\tau_\xi}]$.*

(3) $h_{\nu(\xi)}(a) = c_1$ *if* $a \in H_\rho$ *and* $\xi \geqq \rho$.

(4) $h_{\nu(\xi)}(a) = c_2$ *if* $a \in H_\rho$ *and* $\xi < \rho$.

PROOF. Let h be the limit of $\{h_\nu\}$. By Lemma 5.1(b), h is eventually constant on $[1, \omega^\tau)$. Thus there are constants c_1 and c_2, and a $\beta < \omega^\tau$ such that $h(a) = c_1$ if $\beta \leqq a < \omega^\tau$, and $h(\omega^\tau) = c_2$. To simplify the notation we shall assume that $\beta = 1$ and that $h_\nu(\omega^\tau) = c_2$ for all ν (and not only for $\nu \geqq \nu_0$ for some $\nu_0 < \lambda$). Note that the result follows trivially from Lemma 7.2 if $c_1 = c_2$, so we assume $c_1 \neq c_2$.

Each function h_ν is thus eventually equal to c_2, so find $\beta(\nu) < \omega^\tau$ such that $h_\nu(a) = c_2$ for all $a \geqq \beta(\nu)$, and define

$$f_\nu(a) = \begin{cases} h_\nu(a) - c_1 & \text{if} \quad a \leqq \beta(\nu) \\ h_\nu(a) - c_2 & \text{if} \quad a > \beta(\nu). \end{cases}$$

Clearly f_ν are continuous and converge pointwise to zero, thus by Lemma 7.2, there is a subset H of $[1, \omega^\tau]$ homeomorphic to $[1, \omega^\tau]$, and a family $\mathcal{F} \subset \{f_\nu\}$ of the same cardinality as λ such that $f(a) = 0$ for all $a \in H$ and $f \in \mathcal{F}$. Again, to simplify notation, assume that $\mathcal{F} = \{f_\nu\}$.

We now define H_ξ and $\nu(\xi)$ inductively as follows:

The net $\beta(\nu)$ is unbounded in $[1, \omega^\tau]$ (since $h_\nu(a) \to c_1$ for every $a < \omega^\tau$). We can thus find $\nu(1)$ such that $H \cap [1, \beta(\nu(1))]$ has type which is bigger than τ_1 and choose $H_1 \subset H \cap [1, \beta(\nu(1))]$ homeomorphic to $[1, \omega^{\tau_1}]$.

Inductively, if H_ξ, $\nu(\xi)$ are already chosen for all $\xi < \xi_0$ we let $\bar{\beta} = \sup\{\beta(\xi) \colon \xi < \xi_0\}$, and choose $\nu(\xi_0)$ such that the type of $H \cap [\bar{\beta}, \beta(\nu(\xi_0))]$ is bigger than τ_{ξ_0}, and choose $H_{\xi_0} \subset H \cap [\bar{\beta}, \beta(\nu(\xi_0))]$ homeomorphic to $[1, \omega^{\tau_{\xi_0}}]$.

PROOF OF LEMMA 7.1. The proof will be by induction on η. The case $\eta = 1$ follows from the disjointness lemma in a similar way to its application below and a separate proof will not be given.

We also note that if $B \subset D$ then $A = B^{(\beta)} \subset D^{(\beta)} = E$, and thus the condition $A \subset E$ will be automatically satisfied.

We shall prove (and use for our induction hypothesis) a stronger version of (c), namely, that for every $\delta > 0$, Y can be found satisfying (a), (b) and (c′):

(c′) If $y \in Y$ and $a \in A$, then $|Ty(a) - Ty(b)| \leqq \delta \|y\|$ for all $b \in B \cap [a+1, a^+]$. (Taking $\delta = \varepsilon_1 - \varepsilon$ certainly gives (c).)

Let λ be the cofinality of ω^η ($\lambda = \omega$ if η has countable cofinality or is a successor). Notice that $\omega^{\beta+\eta}$ has the same cofinality λ. Let $\{\eta_\lambda \colon \nu \leqq \lambda\}$ be a nondecreasing net with $\eta_\nu < \eta$ and $\eta_\nu \uparrow \eta$ (or $\eta_\nu = \eta - 1$ if η is a successor).

The proof will consist of three steps. In the first two steps we shall construct certain pairs (E_ν, D_ν) of type (η_ν, β) with $D_\nu \subset D$, and such that the D_ν's are contained in disjoint intervals. To each of these pairs we shall in the last step apply the inductive hypothesis to find Y_ν's and pairs (A_ν, B_ν) satisfying (a), (b) and (c′) (for η_ν and $\delta/3$).

The choice of the (E_ν, D_ν) in the second step will be such that the spaces Y_ν constructed here will satisfy the "disjointness condition" for $\delta/3$ (i.e. $\sup|Tf(d)| \leqq \frac{1}{3}\delta \|f\|$ for all $d \in D_{\nu_0}$ and $f \in \mathrm{sp}\{Y_\nu : \nu \neq \nu_0\}$).

If we consider now the sets $A = \overline{\bigcup A_\nu}$, $B = \overline{\bigcup B_\nu}$ and the space $Y_0 = \overline{\mathrm{sp}}\{Y_\nu\}$, they will satisfy (a), (c′) with $2\delta/3$, and (b), except that $R_A(Y_0)$ is *not* $C_0(A)$ but only $(\Sigma_\nu \oplus C_0(A_\nu))_0$.

In order to find Y with $R_A(Y) = C_0(A)$, we have to add to Y_0 functions $f_\nu \in X$ such that $f_\nu|_{A_\mu} = 0$ if $\mu > \nu$; $f_\nu|_{A_\mu} = 1$ if $\nu \geqq \mu$, and such that if $f \in \mathrm{sp}\{f_\nu\}$ and $a \in A$, then $|Tf(a) - Tf(b)| \leqq \frac{1}{3}\delta \|f\|$ for all $f \in [a+1, a^+] \cap B$.

If we can find such functions then it is obvious that $Y = \overline{\mathrm{sp}}\{Y_0, \{f_\nu\}\}$ will satisfy $R_A(Y) = C_0(A)$ and (a), (b), (c′).

Finding these f_ν's is the first step of the proof. In fact, we first find f_ν's and subsets D'_ν of D such that if we put $E'_\nu = (D'_\nu)^{(\beta)}$ then:

(i) Each (E'_ν, D'_ν) is a pair of type (η_ν, β).

(ii) The D'_ν's are contained in disjoint intervals J_ν.

(iii) $f_\nu|_{D'_\mu} = 0$ if $\mu > \nu$, $f_\nu|_{D'_\mu} = 1$ if $\nu \geqq \mu$.

(iv) If $f \in \mathrm{sp}\{f_\nu\}$, and $a \in E'_{\nu_0}$, then $|Tf(a) - Tf(b)| \leqq \frac{1}{3}\delta \|f\|$ for all $f \in [a+1, a^+] \cap D'_{\nu_0}$.

We then, in the second step, pass to a subset of the ν's (of the same cardinality λ) and to subsets D_ν of the D'_ν's, homeomorphic to the D'_ν's, such that the $\delta/3$-"disjointness condition" will hold. We use these D_ν and $E_\nu = D_\nu^{(\beta)}$ in the last step as we described. We now pass to the details.

Step I. Since ω^η has cofinality λ and by the definition of η_ν, we can find an increasing net $\{\gamma_\nu : \nu < \lambda\}$ such that if we put $D''_\nu = D \cap [\gamma_\nu + 1, \gamma_{\nu+1}]$, $E''_\nu = (D''_\nu)^{(\beta)}$, then (E''_ν, D''_ν) are pairs of type (η_ν, β).

At this point we shall distinguish two cases according to $\lambda = \omega$ or $\lambda > \omega$. (This is the only place in the proof that these two cases are different.)

$\lambda = \omega$. Let g_ν be the unique function in X such that $g_\nu|_{E''_\nu} = 1$, $g_\nu|_{E''_\mu} = 0$ for $\mu \neq \nu$. Since Tg_ν is continuous there is an interval $I_\nu \subset [\gamma_\nu + 1, \gamma_{\nu+1}]$, with $I_\nu \cap D''_\nu$ homeomorphic to D''_ν such that the oscillation of Tg_ν on I_ν is less than $\delta/6$. By applying the disjointness lemma we can now pass to an infinite set M of $\{\nu : \nu < \omega\}$, and find for each $\nu \in M$ a subset D'_ν of $D''_\nu \cap I_\nu$ such that for all

$\nu_0 \in M$, $d \in D'_{\nu_0}$ and $f \in \mathrm{sp}\{g_\nu : \nu \in M, \nu \neq \nu_0\}$ we have $|Tf(d)| \leqq \frac{1}{6}\delta \|f\|$. To simplify notation assume that $M = \{\nu : \nu < \omega\}$ and we then define $f_\nu = \Sigma_{\mu=1}^{\nu} g_\mu$.

These f_ν's and D'_ν's satisfy (i)–(iv) as required.

$\lambda > \omega$. This is where we use Lemma 7.3. Let f_ν be the unique function in X such that $R_E f_\nu = \chi_{E \cap [1, \gamma_\nu]}$. The net $\{\chi_{E \cap [1,\gamma_\nu]}\}$ is a weak Cauchy net in $C_0(E)$, and since $R_E : X \to C_0(E)$ is an isometry onto, $\{f_\nu\}$ is also a weak Cauchy net. Let $h_\nu = Tf_\nu$, then h_ν is a weak Cauchy net in $C_0(\gamma)$, and in particular it is a pointwise convergent net. We shall be interested only in the values of the h_ν's on D which is homeomorphic to $[1, \omega^{\beta+\eta}]$, and $\tau = \beta + \eta$ has cofinality $\lambda > \omega$. Thus by Lemma 7.3 there are constants c_1 and c_2, subsets D'_ν of D and a subnet of $\{h_\nu\}$ (which, to simplify notation, we shall assume is the given net) such that

(1) D'_ν is homeomorphic to $[1, \omega^{\beta+\eta_\nu}]$.

(2) The D_ν's are contained in disjoint intervals.

(3) $h_\nu(a) = c_1$ if $a \in D'_\mu$ and $\nu \geqq \mu$.

(4) $h_\nu(a) = c_2$ if $a \in D'_\mu$ and $\nu < \mu$.

(3) and (4) certainly imply that (iv) holds. In fact, if $f \in \mathrm{sp}\{f_\nu\}$ then Tf is *constant* on each D'_ν hence (iv) holds with zero instead of $\delta/3$.

Step II. We apply the disjointness lemma for $\delta/3$, the sets D'_ν and the spaces $Z_\nu = \{f \in X : f \text{ vanishes on } E \backslash J_\nu\}$.

To simplify notation, assume that the resulting subset of $\{\nu : \nu < \lambda\}$ is the whole set $\{\nu : \nu < \lambda\}$, and let D_ν be the resulting subsets of D'_ν.

Step III. For each ν, put $E_\nu = (D_\nu)^{(\beta)}$. By Lemma 6.2(a) there is an $X_\nu \subset Z_\nu$ with $R_{E_\nu}(X_\nu) = C_0(E_\nu)$ which is determined over (E_ν, D_ν).

We now use the inductive hypothesis for each X_ν, E_ν, D_ν and $\delta/3$, as described in the beginning.

References

1. C. Bessaga and A. Pelczynski, *Spaces of continuous functions IV*, Studia Math. **19** (1960), 53–62.
2. P. Billard, *Sur la primarité des espaces $C(\alpha)$*, to appear in Studia Math.
3. P. G. Casazza, *James' quasi-reflexive space is primary*, to appear.
4. S. P. Gulko and A. V. Oskin, *Isomorphic classification of spaces of continuous functions on totally ordered compact sets*, Functional Anal. Appl. **9** (1975), 56–57.
5. J. Kupka, *A short proof and generalization of a measure theoretic disjointization lemma*, Proc. Amer. Math. Soc. **45** (1974), 70–72.
6. K. Kuratowski, *Topology I*, Academic Press and PWN, Warszawa, 1966.
7. S. V. Kislakov, *Classification of spaces of continuous functions on ordinals*, Siberian Math. J. **16** (1975), 226–231.
8. M. A. Labbé, *Isomorphism of continuous function spaces*, Studia Math. **52**, (1975), 221–231.

9. J. Lindenstrauss and A. Pelczynski, *Contributions to the theory of the classical Banach spaces*, J. Functional Analysis **8** (1971), 225–249.

10. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Lecture notes in Mathematics 338, Springer-Verlag, 1973.

11. S. Mazurkiewicz and W. Sierpinski, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. **1** (1920), 17–27.

12. B. S. Mityagin, *The homotopy structure of the linear group of a Banach space*, Russian Math. Surveys **25** (1970), 59–103.

13. B. S. Mityagin and I. S. Edel'shtein, *Homotopy type of linear groups of two classes of Banach spaces*, Functional Anal. Appl. **4** (1970), 221–231.

14. A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.

15. H. P. Rosenthal, *On relativity disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1970), 13–36.

16. Z. Semadeni, *Banach spaces non-isomorphic to their cartesian squares II*, Bull Acad. Polon. Sci. **8** (1960), 81–84.

17. W. Sierpinski, *Cardinal and Ordinal Numbers*, PWN, Warsaw, 1965.

DEPARTMENT OF MATHEMATICS
OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210 USA